On the convergence of statistical solutions of the 3D Navier-Stokes-α model as α vanishes

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Abstract

In this paper statistical solutions of the 3D Navier-Stokes-α model with periodic boundary condition are considered. It is proved that under certain natural conditions statistical solutions of the 3D Navier-Stokes-α model converge to statistical solutions of the exact 3D Navier-Stokes equations as α goes to zero. The statistical solutions considered here arise as families of time-projections of measures on suitable trajectory spaces.

Keywords. Statistical solutions, Navier-Stokes equations, Navier-Stokes-α model

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1 Introduction

In this work we investigate the convergence of statistical solutions of the three-dimensional Navier-Stokes-α model to statistical solutions of the three-dimensional Navier-Stokes equations, as α goes to zero. We consider the equations with periodic boundary conditions and zero space average, and the statistical solutions are considered in the sense recently introduced by Foias, Rosa and Temam in [25, 23].

Most of the knowledge concerning turbulent flows are rooted in heuristic and phenomenological arguments. One of the fundamental observations is that, although quite irregular, turbulent flows display a certain order in a statistical sense, so that mean quantities and other low order moments are usually more regular and predictable. The statistical solutions that we investigate are mathematical objects used to address the statistical properties of the flow in a rigorous mathematical way directly from the equations of motion.

The Navier-Stokes equations have been widely used as a model for Newtonian turbulent flows (see e.g. [2, 28, 32, 39]) and a rigorous mathematical formalization of the statistical theory of turbulence came with the introduction of statistical solutions for these equations. Several statistical estimates can be obtained rigorously using statistical solutions; see for instance [16, 10, 3, 17, 27, 20, 34]. Therefore, a better understanding of statistical solutions is crucial for a rigorous mathematical theory of turbulence.

The concept of statistical solutions was first introduced by Foias and Prodi [14, 22] in the early 1970's (see also an earlier related mathematical work by Hopf [30]). They considered as a statistical solution a family of measures on the phase space satisfying a Liouville-type equation together with some regularity conditions. Some years later Vishik and Fursikov [43] introduced a different type of statistical solutions, given by measures on suitable trajectory spaces (see also [44, 45]). More recently, in [25, 23], Foias, Rosa and Temam elaborated a notion of statistical solutions which, in some sense, relates the two previous notions and has better analytical properties than the previous ones. More precisely, they considered a Borel probability measure on the space $\mathcal{C}([0,T],H_w)$ which is carried by the trajectory space of Leray-Hopf weak solutions. The family of time projections of such

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a measure gives rise to a statistical solution in the sense of Foias-Prodi that possess additional analytical properties.

We also mention a work due to Capinski and Cutland, [5], preceding the works of Foias, Rosa and Temam, in which the authors prove, using non-standard analysis, the existence of space-time statistical solutions (and of individual weak solutions) defined in the same spirit as that in [25, 23], with the main difference that they use a slightly different definition of weak solutions upon which the statistical solutions are built.

There is also a large literature on the stochastic version of the Navier-Stokes equations, which relies on some ideas from the Vishik-Fursikov formulation; see e.g. [13].

A few other equations have also been considered as suitable models for turbulent fluid motions in specific aspects. For instance, the Navier-Stokes-α model is known as a good approximation model for well-developed turbulent flows (see [6, 7, 8] for the cases of infinite pipes and channels). Also, the 3D Navier-Stokes-Voigt equations are known as an appropriated model for direct numerical simulations of turbulent flows in statistical equilibrium. For a survey on approximating models of the Navier-Stokes equations and their properties we refer the reader to the Introduction given in [29]. Regarding the importance of statistical solutions to yield statistical estimates rigorously, we highlight the work of Ramos and Titi, [35]. The authors derived statistical properties of the invariant measures associated with the solutions of the 3D Navier-Stokes-Voigt equations and established the convergence of probability invariant measures associated with the 3D Navier-Stokes-Voigt equations to stationary statistical solution of the 3D Navier-Stokes equations. With this result, they argue, via statistical estimates obtained rigorously, that the 3D Navier-Stokes-Voigt is in fact a reliable subgrid scale model for direct numerical simulations of turbulent flows.

We focus our study on the Navier-Stokes-α model and on the statistical solution defined by Foias, Rosa and Temam, namely a family of time projections of a measure carried by the trajectory space of the Leray-Hopf weak solutions; see Section 2.6. The Navier-Stokes-α model (also known as Camassa-Holm equations) was introduced by Chen *et al*, in [6]. This model is a regularized approximation of the 3D Navier-Stokes equations such that, in some terms of the equation, the velocity field is replaced by a smoother (filtered) velocity field depending on a small parameter $\alpha > 0$; see Section 2.4. As observed in [18], this regularized approximation introduces an energy penalty that inhibits the creation of smaller and smaller excitations below the length scale α. More precisely, it was proved in [18] that the wavenumber spectrum of the translational kinetic energy for the Navier-Stokes-α model rolls off as κ^{-3} for $\kappa\alpha > 1$ instead of continuing along the Kolmogorov scaling law, $\kappa^{-5/3}$, which is followed for $\kappa\alpha < 1$.

In this article we prove that, under certain natural conditions, statistical solutions of the Navier-Stokes-α model converge to statistical solutions of the Navier-Stokes equations as α goes to zero. We consider both stationary and time-dependent statistical solutions. We point out the importance of stationary statistical solutions in the study of turbulence in statistical equilibrium in time, while, time-dependent statistical solutions are useful in the study of evolving or decaying turbulence.

Another important fact is that the notion of stationary statistical solutions provides a generalization of the notion of invariant measures of semigroups. In the case of the 3D-Navier-Stokes equations this is currently needed since the well-posedness of the equations has not been established. In the two-dimensional case, in which the Navier-Stokes equations have a well-defined semigroup, the two notions have been proved to agree with each other.

The definition of statistical solution for the Navier-Stokes-α model is inspired by the corresponding definition for the Navier-Stokes equations, being the family of projections in time of a Borel probability measure in a suitable trajectory space and carried by the set of individual solutions of the equation. The natural trajectory space for the Navier-Stokes-α model is $\mathcal{C}([0,T],H)$, which is included in $\mathcal{C}([0,T],H_w)$, which is the natural space for the Navier-Stokes equations, so that both statistical solutions can be regarded as projections of measures on

the same space. In this way, both statistical solutions can be viewed as mathematical objects of the same type, allowing a direct comparison between them, and yielding a natural framework for studying the convergence of the corresponding statistical solutions as α goes to zero. Of course, since the Navier-Stokes-α model is well-posed, its solution semigroup induces a measure in the trajectory space, starting from any initial measure in the phase space, in such a way that the time-projections of such a measure form indeed a statistical solution in the sense we consider here (see Section 2.6). It is expected that the converse is also true, namely, that any statistical solution for the Navier-Stokes-α model is the family of projections of a measure induced by the solution semigroup (as it happens for the two-dimensional Navier-Stokes equations), but we do not address this issue here.

This is one of a few results about convergence of statistical solutions (see e.g. [12, 11, 35] for other equations) and it is the first result of convergence for this type of statistical solution, and we believe the main ideas presented here will be valuable for extending the result to other types of approximations (see [4]).

The structure of the paper is as follows. We start with some usual definitions and basic results concerning the functional-analytic framework. In Section 2.2, we review some facts regarding Borel measures and we state some compactness results in the space of Borel measures which may not be so familiar to the reader and which will be used to obtain the convergence of the measures related with the statistical solutions. Later, in Sections 2.3 and 2.4, we briefly introduce the Navier-Stokes equations and the Navier-Stokes-α model, focusing on the results that will be important throughout this paper. In Sections 2.5 and 2.6, we recall the definition of statistical solutions for the Navier-Stokes equations and introduce a notion of statistical solution for the Navier-Stokes-α model.

The main goals of this paper are presented in the final Section 3, where we state and prove the convergence theorems for the measures on the trajectory space (Theorem 3.1 and Corollary 3.3) and for the statistical solutions (Theorem 3.2,Corollary 3.4 and Theorem 3.3). This section is divided into two parts, one for the time-dependent case and the other for the particular case of stationary statistical solutions. The main condition for the convergence of a family of statistical solutions of the α-Navier-Stokes equations to a statistical solution of the Navier-Stokes equations as α goes to zero is that the mean kinetic energy of the family be uniformly bounded. This provides the tightness of the family of measures that guarantees the compactness of the family according to the theory of Topsoe discussed in Section 2.2. The main work, then, is to show that this compactness is sufficient to guarantee that the limit family of measures is a statistical solution of the Navier-Stokes equations. A crucial step in this proof is to show that the limit measure is carried by the space of Leray-Hopf weak solutions of the Navier-Stokes equations. This step depends very much on a result by Vishik, Titi, and Chepyzhov [46] on the convergence of individual solutions.

2 Preliminaries

In this section, we set the notation and provide the definitions and results that are needed throughout this work.

2.1 Functional setting

Let $\Omega := \Pi_{i=1}^3(0, L_i)$, where $L_i > 0$, for $i = 1, 2, 3$, and let $C_{per}^\infty(\Omega; \mathbb{R}^3)$ represent the space of the infinitely differentiable functions $\mathbf{u}$, with $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}), u_3(\mathbf{x})) \in \mathbb{R}^3$ defined for $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, which are Ω-periodic. We define the set of periodic test functions with vanishing average as

$$\mathcal{V} := \{\mathbf{u} \in \mathcal{C}_{per}^\infty(\Omega; \mathbb{R}^3) : \nabla \cdot \mathbf{u} = 0 \text{ and } \int_\Omega \mathbf{u}(\mathbf{x}) \mathrm{d}\mathbf{x} = 0\}.$$

Let H be the closure of $\mathcal{V}$ in $L^2(\Omega; \mathbb{R}^3)$ and let V be the closure of $\mathcal{V}$ in $H^1(\Omega; \mathbb{R}^3)$. The inner product and

the norm in H are defined, respectively, by

$$(\mathbf{u},\mathbf{v}) := \int_\Omega \mathbf{u}\cdot\mathbf{v}\mathrm{d}\mathbf{x} \quad \text{and} \quad |\mathbf{u}| := \sqrt{(\mathbf{u},\mathbf{u})},$$

where $\mathbf{u}\cdot\mathbf{v} = \sum_{i=1}^3 u_i v_i$, and in V they are defined by

$$((\mathbf{u},\mathbf{v})) := (\nabla\mathbf{u},\nabla\mathbf{v}) \quad \text{and} \quad \|\mathbf{u}\| := \sqrt{((\mathbf{u},\mathbf{u}))},$$

where it is understood that $\nabla\mathbf{u} = (\partial u_i/\partial x_j)_{i,j=1}^3$ and that the second term is the integral of the componentwise product between $\nabla\mathbf{u}$ and $\nabla\mathbf{v}$. We also consider the space H endowed with its weak topology, which we denote by H_w.

Let A be the **Stokes operator** defined as $A = -\mathbb{P}\Delta$, where $\mathbb{P} : L^2(\Omega)^3 \to H$ is the Leray-Helmholtz projection, i.e., the orthogonal projector in $L^2(\Omega)^3$ onto the subspace of divergence-free vector fields. We denote by $D(A)$ the domain of A, which is defined as the set of functions $\mathbf{u}\in V$ such that $A\mathbf{u}\in H$. Recall that, in the periodic case with zero space average, $A\mathbf{u} = -\Delta\mathbf{u}$ for $\mathbf{u}\in D(A) = V\cap H^2(\Omega)^3$ and A is a positive self-adjoint operator with compact inverse, so that it has a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ of positive eigenvalues counted according to their multiplicity, in increasing order, associated with an orthonormal basis $\{\mathbf{w}_i\}_{i\in\mathbb{N}}$ in H. Furthermore, the Poincaré inequality holds, i.e., for all $\mathbf{u}\in V$,

$$\lambda_1|\mathbf{u}|^2 \le \|\mathbf{u}\|^2, \tag{1}$$

where $\lambda_1 > 0$ is the first eigenvalue of the Stokes operator.

Denote by P_k the **Galerkin projector** defined as the projector onto the space spanned by the eigenfunctions associated with the first k eigenvalues, i.e.,

$$P_k\mathbf{u} = \sum_{i=1}^k (\mathbf{u},\mathbf{w}_i)\mathbf{w}_i, \ \forall \mathbf{u}\in H.$$

There are some important properties of P_k that we want to highlight. For instance, we have that, for all $\mathbf{u}\in H$ and for all $k\in\mathbb{N}$, $|P_k\mathbf{u}| \le |P_{k+1}\mathbf{u}|$ and $|P_k\mathbf{u}|\le|\mathbf{u}|$. We also have that $P_k : H_w \to H$ is continuous. Indeed, observe that the open sets in H can be characterized by a basis of neighborhoods

$$\mathcal{O}(\mathbf{u},r) = \{\mathbf{w}\in H : |\mathbf{u}-\mathbf{w}| < r\},$$

for $\mathbf{u}\in H$ and $r>0$. On the other hand, the collection of open sets in H_w has a characterization by a basis of neighborhoods given by

$$\mathcal{O}_w(\mathbf{u},r,\mathbf{v}_1,\dots,\mathbf{v}_N) = \{\mathbf{w}\in H_w : \sum_{i=1}^N |(\mathbf{u}-\mathbf{w},\mathbf{v}_i)|^2 < r^2\},$$

for $\mathbf{u}\in H_w$, $r>0$, $N\in\mathbb{N}$, and $\mathbf{v}_1,\dots,\mathbf{v}_N\in H$. Notice that if we prove that for all $\mathbf{u}\in H_w$ and $r>0$ the set $P_k^{-1}(\mathcal{O}(P_k\mathbf{u},r))$ is open in H_w, then we obtain that $P_k : H_w\to H$ is continuous. This follows from the fact that $P_k^{-1}(\mathcal{O}(P_k\mathbf{u},r))$ is of the form

$$\begin{aligned} P_k^{-1}(\mathcal{O}(P_k\mathbf{u},r)) &= \{\mathbf{w}\in H_w : P_k\mathbf{w}\in\mathcal{O}(P_k\mathbf{u},r)\} = \{\mathbf{w}\in H_w : |P_k\mathbf{u}-P_k\mathbf{w}| < r\} \\ &= \{\mathbf{w}\in H_w : \sum_{i=1}^k |(\mathbf{u}-\mathbf{w},\mathbf{w}_i)|^2 < r^2\}, \end{aligned}$$

and, hence, is an element of the basis for H_w.

For the functional setting concerning the Navier-Stokes-α model, we adopt the framework introduced by Vishik, Titi and Chepyzhov in [46].

We start with the natural space for the solutions of the Navier-Stokes-α model. Given an interval $I \subset \mathbb{R}$, we define

$$\mathcal{F}_I = \{\mathbf{z} : \mathbf{z}(\cdot) \in L^2_{loc}(I;V) \cap L^\infty_{loc}(I;H), \partial_t \mathbf{z}(\cdot) \in L^2_{loc}(I;D(A)')\}. \tag{2}$$

We endow this space with its natural weak-type topology, which we term τ **topology**, and which can be defined in terms of nets as follows: a net of functions $\{\mathbf{z}_\gamma\}_\gamma \subset \mathcal{F}_I$ converges to a function $\mathbf{z} \in \mathcal{F}_I$ in the topology τ if for each compact interval $J \subset I$,

$$\mathbf{z}_\gamma \overset{*}{\rightharpoonup} \mathbf{z} \text{ in } L^\infty(J;H), \quad \mathbf{z}_\gamma \rightharpoonup \mathbf{z} \text{ in } L^2(J;V), \quad \text{and} \quad \partial_t \mathbf{z}_\gamma \rightharpoonup \partial_t \mathbf{z} \text{ in } L^2(J;D(A)').$$

Consider also the following Banach space

$$\mathcal{F}_I^b = \{\mathbf{z} : \mathbf{z}(\cdot) \in L^2_b(I;V) \cap L^\infty(I;H), \partial_t \mathbf{z}(\cdot) \in L^2_b(I;D(A)')\}, \tag{3}$$

with norm given by

$$\|\mathbf{z}\|_{\mathcal{F}_b} = \|\mathbf{z}\|_{L^2_b(I,V)} + \|\mathbf{z}\|_{L^\infty(I,H)} + \|\partial_t \mathbf{z}\|_{L^2_b(I,D(A)')},$$

where

$$\|\mathbf{z}\|_{L^2_b(I,V)} = \sup_{\{t \in I : t+1 \in I\}} \int_t^{t+1} \|\mathbf{z}(s)\|^2 ds,$$

and

$$\|\partial_t \mathbf{z}\|_{L^2_b(I,D(A)')} = \sup_{\{t \in I : t+1 \in I\}} \int_t^{t+1} \|\partial_t \mathbf{z}(s)\|^2_{D(A)'} ds.$$

Finally, we introduce a natural space for the solutions of the Navier-Stokes equations, which is the space $\mathcal{C}_{loc}(I, H_w)$ of continuous functions from an interval $I \subset \mathbb{R}$ to H_w, where, as defined above, H_w stands for the space H endowed with the weak topology. This space can also be seen as the space of weakly continuous function from I to H. The topology on $\mathcal{C}_{loc}(I, H_w)$ is that of uniform convergence in H_w on compact intervals of I.

2.2 Measures and Compactness on the space of measures

The natural measure space in our framework is the space of Borel probability measures on $\mathcal{C}_{loc}(I, H_w)$. Since we are interested in the study of the convergence of family of measures we need a compactness result on the space of measures. For measures on a separable metrizable space there is the well-known compactness result due to Prohorov [33]. In our case, however, $\mathcal{C}_{loc}(I, H_w)$ is not metrizable, so we use a more general compactness result due to Topsoe (see [40, 41, 42]), which is suitable for our framework.

Let X be a Hausdorff space and let $\mathcal{B}_X$ be the Borel σ-algebra on X. The set of finite Borel measures on X is denoted by $\mathcal{M}(X)$ and the set of Borel probability measures is denoted by $\mathcal{P}(X)$. We say that a measure μ is **tight** if for every set $A \in \mathcal{B}_X$,

$$\mu(A) = \sup\{\mu(K) : K \text{ is compact, } K \subset A\}.$$

Denote by $\mathcal{M}(X;t)$ the set of all Borel finite measures which are tight, and by $\mathcal{P}(X;t)$ the set of all measures in $\mathcal{P}(X)$ which are tight. If X is a Polish space (i.e., a separable and completely metrizable space) then every finite Borel measure is tight, so that $\mathcal{M}(X;t) = \mathcal{M}(X)$ and $\mathcal{P}(X;t) = \mathcal{P}(X)$.

We say that a net (μ_γ) in $\mathcal{P}(X)$ is **uniformly tight** if for every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that

$$\mu_\gamma(X \setminus K) < \varepsilon, \quad \forall \gamma.$$

Let Y be a Hausdorff space and let $F : X \to Y$ be a continuous function. For a measure $\mu \in \mathcal{M}(X)$ we define the measure $F\mu$ induced by μ on Y as $F\mu(E) = \mu(F^{-1}(E))$, for every Borel set $E \subset Y$. In this setting, the Change of Variables Theorem (see e.g. [1]) says that if $\varphi : Y \to \mathbb{R}$ is a $F\mu$-integrable function then $\varphi \circ F$ is μ-integrable and

$$\int_X \varphi(F(x))d\mu(x) = \int_Y \varphi(y)dF\mu(y). \tag{4}$$

Now, for the result of compactness on the space of measures we endow the space $\mathcal{M}(X)$ with the weakest topology for which the mapping $\mu \mapsto \mu(f)$ is upper semicontinuous for every f bounded, real-valued, upper semicontinuous function on X, where $\mu(f)$ stands for the integral $\int_X f(x)d\mu(x)$. We use the symbol $\overset{w}{\to}$ to denote the convergence of nets in $\mathcal{M}(X)$ with respect to this weak topology. The spaces $\mathcal{P}(X)$, $\mathcal{M}(X;t)$, and $\mathcal{P}(X;t)$ are endowed with the topology inherited from $\mathcal{M}(X)$.

Recall that a topological space X is **completely regular** if every nonempty closed set and every singleton disjoint from it can be separated by a continuous function.

In [40], Topsoe proved a result of compactness on the space of measures on an abstract space X. In the case X is a topological Haudorff space the result is reduced to the following(see [41, Theorem 9.1]):

Theorem 2.1. *Let X be a Hausdorff space. Let (μ_γ) be a net in $\mathcal{P}(X;t)$ which is uniformly tight. Then, there exist $\mu \in \mathcal{P}(X;t)$ and a subnet (μ_{γ_β}) such that $\mu_{\gamma_\beta} \overset{w}{\to} \mu$.*

Topsoe also proved, in [41], the following result:

Lemma 2.1. *Let X be a completely regular Hausdorff space. For a net (μ_γ) in $\mathcal{M}(X)$ and $\mu \in \mathcal{M}(X,t)$, the following statements are equivalent*

(1) $\mu_\gamma \overset{w}{\to} \mu$;

(2) $\limsup \mu_\gamma(f) \leq \mu(f)$, *for all f bounded upper semicontinuous function;*

(3) $\liminf \mu_\gamma(f) \geq \mu(f)$, *for all f bounded lower semicontinuous function;*

(4) $\lim \mu_\gamma(f) = \mu(f)$, *for all bounded continuous function f.*

Remark 2.1. *The last lemma was actually stated in a more general setting. For instance, if X is only Hausdorff and $\mu \in \mathcal{M}(X)$ then the first three statements are equivalent and each of them implies the last one. If X is a completely regular Hausdorff space and $\mu \in \mathcal{M}(X)$ is τ-smooth (a condition which is satisfied for every tight measure) then all the statements are equivalent.*

Observe that, by Theorem 2.1, if (μ_γ) is a net in $\mathcal{P}(X;t)$ which is uniformly tight then there exists a subnet that converges to a limit μ which is tight. Moreover, by Lemma 2.1, if X is a completely regular Hausdorff space, the convergence in the weak topology in $\mathcal{P}(X;t)$ is equivalent to the usual convergence $\mu_{\gamma_\beta}(f) \to \mu(f)$, for every $f \in \mathcal{C}_b(X)$, denoted by $\mu_{\gamma_\beta} \overset{*}{\rightharpoonup} \mu$, where $\mathcal{C}_b(X)$ denotes the set of bounded, continuous, real-valued functions defined on X.

We state the last observation as the next theorem:

Theorem 2.2. *Let X be a completely regular Hausdorff space. Let (μ_γ) be a net in $\mathcal{P}(X;t)$ which is uniformly tight. Then, there exist $\mu \in \mathcal{P}(X;t)$ and a subnet (μ_{γ_β}) such that $\mu_{\gamma_\beta} \overset{*}{\rightharpoonup} \mu$, i.e.,*

$$\lim_\beta \mu_{\gamma_\beta}(f) = \mu(f), \qquad \textit{for all } f \in \mathcal{C}_b(X).$$

Also in [41], Topsoe stated the following result, which we prove here with more details.

Theorem 2.3. *Let X be a Hausdorff space. Then, $\mathcal{P}(X;t)$ is a Hausdorff space.*

Proof. First, recall that a Hausdorff space can be characterized as a topological space where every net converges to at most one point. Therefore, it is enough to prove that if $(\mu_\gamma)_\gamma$ is a net in $\mathcal{P}(X;t)$ which converges to two measures $\mu_1, \mu_2 \in \mathcal{P}(X;t)$, i.e., $\mu_\gamma \overset{w}{\to} \mu_1$ and $\mu_\gamma \overset{w}{\to} \mu_2$, then $\mu_1 = \mu_2$. Let $A \in \mathcal{B}_X$, denote by $\mathring{A}$ the interior of A and by $\bar{A}$ the closure of A. It is clear that the characteristic functions $\chi_{\mathring{A}}$ and $\chi_{\bar{A}}$ are, respectively, lower semicontinuous and upper semicontinuous functions. Therefore, using Lemma 2.1, we obtain that

$$\mu_1(\mathring{A}) \le \liminf_\gamma \mu_\gamma(\mathring{A}) \le \limsup_\gamma \mu_\gamma(\bar{A}) \le \mu_2(\bar{A}).$$

Now, let $E \in \mathcal{B}_X$, and let us prove that $\mu_1(E) \le \mu_2(E)$. In order to do so, consider any compact sets $K_1 \subset E$ and $K_2 \subset E^c$. Since X is Hausdorff there exist disjoint open sets A and B such that such that $K_1 \subset A$ and $K_2 \subset B$. It is clear that $\bar{A} \subset X \setminus K_2$. Thus,

$$\mu_1(K_1) \le \mu_1(A) \le \mu_2(\bar{A}) \le \mu_2(X \setminus K_2) = 1 - \mu_2(K_2),$$

which leads us to

$$\mu_1(K_1) + \mu_2(K_2) \le 1.$$

Since K_1 and K_2 are arbitrary compact sets satisfying $K_1 \subset E$ and $K_2 \subset E^c$, we can take the supremum over all compact sets $K_1 \subset E$ and the supremum over all compact sets $K_2 \subset E^c$ in the last expression, and we find that

$$\sup\{\mu_1(K_1) : K_1 \text{ is compact}, K_1 \subset E\} + \sup\{\mu_1(K_2) : K_2 \text{ is compact}, K_2 \subset E^c\} \le 1.$$

Since μ_1 and μ_2 are tight, we conclude that

$$\mu_1(E) + \mu_2(E^c) \le 1.$$

Thus $\mu_1(E) \le \mu_2(E)$, for all $E \in \mathcal{B}_X$. Now, since $\mu_1(X) = \mu_2(X) = 1$ it follows that $\mu_1 = \mu_2$. □

Let X be a completely regular Hausdorff space and $\mu_1, \mu_2 \in \mathcal{P}(X;t)$. Then,

$$\mu_1 = \mu_2 \text{ if and only if } \int_X \varphi(x)\mathrm{d}\mu_1(x) = \int_X \varphi(x)\mathrm{d}\mu_2(x), \forall \varphi \in \mathcal{C}_b(X). \tag{5}$$

Indeed, suppose that $\int_X \varphi(x)\mathrm{d}\mu_1(x) = \int_X \varphi(x)\mathrm{d}\mu_2(x)$, for all $\varphi \in \mathcal{C}_b(X)$. We define the net $(\mu_\gamma)_\gamma$, where $\mu_\gamma = \mu_2$, for all γ. Then, it is clear that $\mu_\gamma \overset{w}{\to} \mu_1$ and $\mu_\gamma \overset{w}{\to} \mu_2$. Since $\mathcal{P}(X;t)$ is Hausdorff we find that $\mu_1 = \mu_2$. The other implication is trivial.

2.3 The Navier-Stokes equations

We state only the results, properties and estimates that are needed throughout this work. For a more complete theory of the Navier-Stokes equations the reader is referred to [9, 18, 31, 37, 38], and the references therein.

In the estimates below, we will consider certain quantities which are constant with respect to the solutions of the equations, but which may depend on the coefficients of the equations, the forcing terms and the spatial domain. We will call them non-dimensional constants when they are independent of re-scalings of the equations in space and time, hence, in particular, they may depend on the shape of the domain, but not on the size of the domain. Moreover, a non-dimensional constant will be called universal when it does not depend on any of the parameters of the equations, not even on the shape of the domain.

We recall that the Navier-Stokes equations can be written in the following functional form

$$\mathbf{u}_t + \nu A\mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \mathbf{f}, \tag{6}$$

where A is the Stokes operator and $B(\mathbf{u}, \mathbf{u}) = \mathbb{P}[(\mathbf{u} \cdot \nabla)\mathbf{u}]$, for $\mathbf{u} \in V$.

The notion of solution that is considered here is the well-known Leray-Hopf weak solution, which is defined below.

Definition 2.1. *Let $\mathbf{f} \in L^2_{loc}(I, V')$. A function $\mathbf{u}$ is called a* ***Leray-Hopf weak solution*** *if:*

(i) $\mathbf{u} \in L^\infty_{loc}(I, H) \cap L^2_{loc}(I, V) \cap \mathcal{C}_{loc}(I, H_w)$;

(ii) $\partial_t \mathbf{u} \in L^{4/3}_{loc}(I, V')$;

(iii) $\mathbf{u}$ *satisfies the weak formulation of the Navier-Stokes equations, i.e.,*

$$\mathbf{u}_t + \nu A\mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \mathbf{f},$$

in V', in the sense of distributions on I;

(iv) $\mathbf{u}$ *satisfies the energy inequality in the sense that for almost all $t' \in I$ and for all $t \in I$ with $t > t'$,*

$$\frac{1}{2}|\mathbf{u}(t)|^2 + \nu \int_{t'}^{t} \|\mathbf{u}(s)\|^2 \mathrm{d}s \le \frac{1}{2}|\mathbf{u}(t')|^2 + \int_{t'}^{t} (\mathbf{f}(s), \mathbf{u}(s))\mathrm{d}s; \tag{7}$$

(v) If I is closed and bounded on the left, with left end point t_0, then the solution is strongly continuous in H at t_0 from the right, i.e., $\mathbf{u}(t) \to \mathbf{u}(t_0)$ in H as $t \to t_0^+$.

The set of allowed times t' in (7) can be characterized as the points of strong continuity of $\mathbf{u}$, in H, from the right. In particular, condition (v) implies that $t' = t_0$ is allowed in that case.

Suppose that $\mathbf{f} \in L^\infty(I, H)$ and let $\mathbf{u}$ be a Leray-Hopf weak solution. It is known that (see e.g. [21, Appendix II.B.1])

$$|\mathbf{u}(t)|^2 \le |\mathbf{u}(t')|^2 e^{-\lambda_1 \nu (t-t')} + \frac{1}{\lambda_1^2 \nu^2} \|\mathbf{f}\|^2_{L^\infty(t',t;H)} (1 - e^{-\lambda_1 \nu (t-t')}), \tag{8}$$

for all $t' \in I$ allowed in (7) and for all $t \in I$ with $t > t'$.

Furthermore, for all $t' \in I$ allowed in (7) and for all $t \in I$ with $t > t'$,

$$\left(\int_{t'}^{t} \|\mathbf{u}(s)\|^2 ds \right)^{1/2} \le \frac{1}{\nu^{1/2}} |\mathbf{u}(t')| + \lambda_1^{1/4} \nu M_1 (t - t')^{1/2}, \tag{9}$$

and

$$\left(\int_{t'}^{t} \|\partial_t \mathbf{u}(s)\|^2_{D(A)'} ds \right)^{1/2} \le \frac{c_1}{\lambda_1^{1/4} \nu^{1/2}} |\mathbf{u}(t')|^2 + \frac{\nu^{3/2}}{\lambda_1^{3/4}} M_1 + \nu^{5/2} \lambda_1^{1/4} M_1 (t - t'), \tag{10}$$

where c_1 is a universal constant and M_1 is a non-dimensional constant which depends only on non-dimensional combinations of the parameters ν, λ_1 and $\|\mathbf{f}\|_{L^\infty(I,H)}$.

Define

$$R_0 := \frac{1}{\lambda_1 \nu} \|\mathbf{f}\|_{L^\infty(I,H)} \tag{11}$$

and observe that if $|\mathbf{u}(t')| \le R$, for some $R \ge R_0$ and some $t' \in I$ allowed in (7), then it follows from (8) that $|\mathbf{u}(t)| \le R$ for all $t \ge t'$.

2.4 The Navier-Stokes-α model

Most of the definitions, properties and results described below were taken from [46].

We consider the 3D Navier-Stokes-α model, in the periodic domain Ω:

$$\begin{cases} \mathbf{v}_t - \nu\Delta\mathbf{v} + \mathbf{u}\times(\nabla\times\mathbf{v}) + \nabla p = \mathbf{f}, \\ \mathbf{v} = \mathbf{u} - \alpha^2\Delta\mathbf{u}, \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{12}$$

where $\mathbf{u}$ is the unknown (filtered) velocity field, $\mathbf{v}$ is an auxiliary variable, p is the pressure, $\mathbf{f}$ is the external force, and $\alpha > 0$ is a constant.

A functional formulation of (12) can be written as

$$\frac{d}{dt}(\mathbf{u} + \alpha^2 A\mathbf{u}) + \nu A(\mathbf{u} + \alpha^2 A\mathbf{u}) + \tilde{B}(\mathbf{u}, \mathbf{u} + \alpha^2 A\mathbf{u}) = \mathbf{f}, \tag{13}$$

where A is the Stokes operator, and $\tilde{B}(\mathbf{u},\mathbf{v}) := \mathbb{P}[\mathbf{u}\times(\nabla\times\mathbf{v})]$ is defined for $\mathbf{u},\mathbf{v}\in V$, with values in V', and which can be extended continuously to an operator from $(\mathbf{u},\mathbf{v})\in V\times H$ with values in $D(A)'$ (see [19]).

The notion of solution for the Navier-Stokes-α model that we consider is stated in the next definition:

Definition 2.2. *Let $\mathbf{f}\in L^2_{loc}(I,H)$. A function $\mathbf{u}$ is a solution of (12) on I if*

(i) $\mathbf{u}\in L^\infty_{loc}(I;V)\cap L^2_{loc}(I;D(A))$;

(ii) $\partial_t\mathbf{u}\in L^2_{loc}(I;H)$;

(iii) $\mathbf{u}\in\mathcal{C}_{loc}(I;V)$;

(iv) $\mathbf{u}$ *satisfies*

$$\frac{d}{dt}(\mathbf{u} + \alpha^2 A\mathbf{u}) + \nu A(\mathbf{u} + \alpha^2 A\mathbf{u}) + \tilde{B}(\mathbf{u}, \mathbf{u} + \alpha^2 A\mathbf{u}) = \mathbf{f}$$

in $D(A)'$, in the sense of distributions on I;

(v) $\mathbf{u}$ *satisfies the energy equality in the sense that for all $t', t\in I$ with $t > t'$,*

$$\begin{aligned} \frac{1}{2}(|\mathbf{u}(t)|^2 + \alpha^2\|\mathbf{u}(t)\|^2) + \nu\int_{t'}^{t}(\|\mathbf{u}(s)\|^2 + \alpha^2|A\mathbf{u}(s)|^2)ds \\ = \frac{1}{2}(|\mathbf{u}(t')|^2 + \alpha^2\|\mathbf{u}(t')\|^2) + \int_{t'}^{t}(\mathbf{f}(s),\mathbf{u}(s))ds. \end{aligned} \tag{14}$$

Observe that conditions (ii), (iii) and (v) are consequences of (i) and (iv).

The existence and uniqueness theorem of solution to the Navier-Stokes-α model was proved in [19]:

Theorem 2.4. *Let $f\in H$ and $\mathbf{u}_0\in V$. Then, for each $T > 0$, there exists a unique solution of (12) on $[0,T]$ in the sense of Definition 2.2 with initial data $\mathbf{u}_0$.*

It is not hard to see that, in fact, one can assume $\mathbf{f}\in L^\infty(I,H)$ and still have the same conclusion as in the previous theorem.

Another important property, which was observed and used in [46] is the following: If $\mathbf{u}$ is a solution of the Navier-Stokes-α model on I, then $\mathbf{w}$, defined by $\mathbf{w} = (1+\alpha^2 A)^{1/2}\mathbf{u}$, satisfies the functional equation

$$\mathbf{w}_t + \nu A\mathbf{w} + (1+\alpha^2 A)^{-1/2}\tilde{B}((1+\alpha^2 A)^{-1/2}\mathbf{w}, (1+\alpha^2 A)^{1/2}\mathbf{w}) = (1+\alpha^2 A)^{-1/2}\mathbf{f}, \tag{15}$$

and vice-versa.

Therefore, based on the definition of solution to the equation (12), a notion of solution for the equation (15) can be defined as follows:

Definition 2.3. *Let $\mathbf{f} \in L^2_{loc}(I;H)$. A function $\mathbf{w}$ is a solution of (15) on I if:*

(i) $\mathbf{w} \in L^\infty_{loc}(I;H) \cap L^2_{loc}(I;V)$;

(ii) $\partial_t \mathbf{w} \in L^2_{loc}(I;D(A)')$;

(iii) $\mathbf{w} \in \mathcal{C}_{loc}(I;H)$;

(iv) $\mathbf{w}$ *satisfies*

$$\mathbf{w}_t + \nu A\mathbf{w} + (1+\alpha^2 A)^{-1/2}\tilde{B}((1+\alpha^2 A)^{-1/2}\mathbf{w}, (1+\alpha^2 A)^{1/2}\mathbf{w}) = (1+\alpha^2 A)^{-1/2}\mathbf{f},$$

in $D(A)'$, in the sense of distributions on I;

(v) $\mathbf{w}$ *satisfies the energy equality in the sense that for all $t', t \in I$ with $t > t'$,*

$$\frac{1}{2}|\mathbf{w}(t)|^2 + \nu\int_{t'}^{t} \|\mathbf{w}(s)\|^2 ds = \frac{1}{2}|\mathbf{w}(t')|^2 + \int_{t'}^{t} ((1+\alpha^2 A)^{-1/2}\mathbf{f}(s), \mathbf{w}(s))ds. \tag{16}$$

Again, here we have that conditions (ii), (iii) and (v) are consequences of (i) and (iv).

Suppose that $\mathbf{f} \in L^\infty(I,H)$. We observe that if $\mathbf{w}$ is a solution of the Navier-Stokes-α model on an interval I, in the sense of Definition 2.3, then $|\mathbf{w}(\cdot)|^2$ is an absolutely continuous function on I (see [46, Corollary 2.1]). As a consequence, one can show that, for any $\psi : [0,\infty) \to \mathbb{R}$, such that $\psi \in \mathcal{C}^1([0,\infty))$, $\psi \geq 0$, $\sup_{r\geq 0}\psi'(r) < \infty$, $\mathbf{w}$ satisfies the following estimate,

$$\psi(|\mathbf{w}(t)|^2) \leq \psi(|\mathbf{w}(t')|^2) + \frac{1}{\lambda_1\nu}\|\mathbf{f}\|_{L^\infty(I,H)} \sup_{r\geq 0}\psi'(r)(t-t'), \tag{17}$$

for all $t, t' \in I$, with $t > t'$.

We also have the following *a priori* estimates for any solution $\mathbf{w}$ of the Navier-Stokes-α model in the sense of Definition 2.3 (adapted from [46, Corollary 3.2]):

$$|\mathbf{w}(t)|^2 \leq |\mathbf{w}(t')|^2 e^{-\nu\lambda_1(t-t')} + \frac{1}{\lambda_1^2\nu^2}\|\mathbf{f}\|^2_{L^\infty(t',t;H)}(1-e^{-\nu\lambda_1(t-t')}), \tag{18}$$

$$\left(\int_{t'}^{t} \|\mathbf{w}(s)\|^2 ds\right)^{1/2} \leq \frac{1}{\nu^{1/2}}|\mathbf{w}(t')| + \lambda_1^{1/4}\nu M_2 (t-t')^{1/2}, \tag{19}$$

$$\left(\int_{t'}^{t} \|\partial_t\mathbf{w}(s)\|^2_{D(A)'} ds\right)^{1/2} \leq \frac{c_2}{\lambda_1^{1/4}\nu^{1/2}}|\mathbf{w}(t')|^2 + \frac{\nu^{3/2}}{\lambda_1^{3/4}}M_2 + \nu^{5/2}\lambda_1^{1/4}M_2(t-t'), \tag{20}$$

for all $t', t \in I$, with $t > t'$, where c_2 is a universal constant and M_2 is a non-dimensional constant which depends only on non-dimensional combinations of the parameters ν, λ_1 and $\|\mathbf{f}\|_{L^\infty(I,H)}$.

Again, we observe that if $|\mathbf{w}(t')| \leq R$, for some $R \geq R_0$, where R_0 is given by (11), and for some $t' \in I$, then it follows from (18) that $|\mathbf{w}(t)| \leq R$ for all $t \geq t'$.

In [46, Theorem 3.1], Vishik, Titi and Chepyzhov proved the convergence of solutions of the Navier-Stokes-α model to solutions of the Navier-Stokes equations. This result was proved in the case where $f \in H$ and $I = [0,\infty)$, which was of interest to them. However, it is not hard to see that the proof can be adapted to the case when I is an arbitrary interval and $f \in L^\infty(I,H)$. Since this result is going to play an important role in this article and for the reader's convenience we state it below:

Theorem 2.5. *Let $I \subset \mathbb{R}$ be an arbitrary interval and $f \in L^\infty(I,H)$. Let $\{\mathbf{w}_n\}$ be a bounded sequence in $\mathcal{F}^b_I$ such that each $\mathbf{w}_n$ is a solution of the Navier-Stokes-α_n model on I, with $\alpha_n \to 0$ as $n \to \infty$, and $\mathbf{w}_n \to \mathbf{w}$ in τ as $n \to \infty$, for some $\mathbf{w} \in \mathcal{F}^b_I$. Then $\mathbf{w}$ is a Leray-Hopf weak solution of the 3D Navier-Stokes equations on $\mathring{I}$.*

2.5 Trajectory spaces

The trajectory spaces made of solutions of the Navier-Stokes equations and of the Navier-Stokes-α model play an important role in the definition of the Vishik-Fursikov measure (see Definitions 2.4 and 2.7) since they connect this notion of solution with the corresponding equations.

Let $I \subset \mathbb{R}$ be an arbitrary interval and $R > 0$. Consider the spaces $\mathcal{C}_{loc}(I, H_w)$ and $\mathcal{C}_{loc}(I, B_H(R)_w)$ endowed with the topology of uniform weak convergence on compact intervals in I. Then, $\mathcal{C}_{loc}(I, H_w)$ is a separable Hausdorff locally convex topological vector space, hence completely regular, and $\mathcal{C}_{loc}(I, B_H(R)_w)$ is a Polish space, that is, a separable and completely metrizable space.

For any interval $J \subset I$, we introduce the restriction operator

$$\begin{aligned} \Pi_J : \mathcal{C}_{loc}(I, H_w) &\to \mathcal{C}_{loc}(J, H_w) \\ \mathbf{u} &\mapsto (\Pi_J \mathbf{u})(t) = \mathbf{u}(t), \ \forall t \in J. \end{aligned}$$

It is clear that the restriction operator is continuous. Furthermore, if J is a closed subinterval of I, then Π_J is also surjective and open.

For each interval I in $\mathbb{R}$ and each $t \in I$ we define the projection operator Π_t by

$$\begin{aligned} \Pi_t : \mathcal{C}_{loc}(I, H_w) &\to H_w \\ \mathbf{u} &\mapsto \Pi_t \mathbf{u} = \mathbf{u}(t). \end{aligned}$$

which is also continuous, surjective and open.

For the Navier-Stokes equations, we define the following trajectory spaces based on the Leray-Hopf weak solutions given by Definition 2.1:

$$\mathcal{U}_I = \{\mathbf{u} \in \mathcal{C}_{loc}(I; H_w) : \mathbf{u} \text{ is a Leray-Hopf weak solution on } I\}, \tag{21}$$

$$\mathcal{U}_I^\sharp = \{\mathbf{u} \in \mathcal{C}_{loc}(I; H_w) : \mathbf{u} \text{ is a Leray-Hopf weak solution on } \mathring{I}\}, \tag{22}$$

where $\mathring{I}$ represents the interior of I. We endow these spaces with the topology inherited from $\mathcal{C}_{loc}(I, H_w)$.

The relation between the two spaces defined above is that $\mathcal{U}_I^\sharp$ is the sequential closure of $\mathcal{U}_I$ with respect to the topology inherited from $\mathcal{C}_{loc}(I; H_w)$. Furthermore it is clear that if I is open then $\mathcal{U}_I = \mathcal{U}_I^\sharp$. The difference appears when I is closed and bounded on the left, since we do not know, in general, whether the weak solutions in $\mathcal{U}_I^\sharp$ are strongly continuous in H, from the right, at the left end point of the interval, as are those in $\mathcal{U}_I$.

Sometimes it will be useful to work with the following spaces

$$\mathcal{U}_I(R) = \{\mathbf{u} \in \mathcal{C}_{loc}(I; B_H(R)_w) : \mathbf{u} \text{ is a Leray-Hopf weak solution on } I\}, \tag{23}$$

$$\mathcal{U}_I^\sharp(R) = \{\mathbf{u} \in \mathcal{C}_{loc}(I; B_H(R)_w) : \mathbf{u} \text{ is a Leray-Hopf weak solution on } \mathring{I}\}, \tag{24}$$

with the topology inherited from $\mathcal{C}_{loc}(I; H_w)$.

As observed in [25], if I is an interval closed and bounded on the left, then, for any sequence $\{R_i\}_{i=1}^\infty$ of positive numbers with $R_i \geq R_0$, for all $i \in \mathbb{N}$, and $R_i \to \infty$, we have the representation

$$\mathcal{U}_I^\sharp = \bigcup_{i=1}^\infty \mathcal{U}_I^\sharp(R_i). \tag{25}$$

And, if I is an interval open on the left, then, for any sequence $\{R_i\}_{i=1}^\infty$ of positive numbers with $R_i \geq R_0$, for all $i \in \mathbb{N}$, and $R_i \to \infty$ and for any sequence $\{J_n\}_{n=1}^\infty$ of compact intervals in I such that $\cup_{n=1}^\infty J_n = I$, we have the representation

$$\mathcal{U}_I^\sharp = \bigcap_{n=1}^\infty \bigcup_{i=1}^\infty \Pi_{J_n}^{-1}\mathcal{U}_{J_n}^\sharp(R_i). \tag{26}$$

As proved in [25], the spaces $\mathcal{U}_I$, $\mathcal{U}_I^\sharp$, $\mathcal{U}_I(R)$ and $\mathcal{U}_I^\sharp(R)$ are all Borel subsets of $\mathcal{C}_{loc}(I,H_w)$ and, in particular, $\mathcal{U}_I^\sharp(R)$ is closed.

For the Navier-Stokes-α model, we consider the solutions in the sense of Definition 2.3. Since these solutions belong to $C_{loc}(I;H)$, we define the following trajectory spaces

$$\mathcal{U}_I^\alpha = \{\mathbf{u} \in \mathcal{C}_{loc}(I;H) : \mathbf{u} \text{ is a solution of the Navier-Stokes-}\alpha \text{ model on } I\}, \tag{27}$$

$$\mathcal{U}_I^\alpha(R) = \{\mathbf{u} \in \mathcal{C}_{loc}(I;B_H(R)) : \mathbf{u} \text{ is a solution of the Navier-Stokes-}\alpha \text{ model on } I\}. \tag{28}$$

In order to compare with the solutions of the Navier-Stokes equations and since $\mathcal{C}_{loc}(I,H)$ is included in $\mathcal{C}_{loc}(I,H_w)$, we shall consider the spaces $\mathcal{U}_I^\alpha$ and $\mathcal{U}_I^\alpha(R)$ with the topology inherited from $\mathcal{C}_{loc}(I,H_w)$.

Here, we also have the same characterizations as the ones for the Navier-Stokes trajectory space. That is, if I is an interval closed and bounded on the left, then, for any sequence $\{R_i\}_{i=1}^\infty$ of positive numbers with $R_i \geq R_0$, for all $i \in \mathbb{N}$, and $R_i \to \infty$, we have the representation

$$\mathcal{U}_I^\alpha = \bigcup_{i=1}^\infty \mathcal{U}_I^\alpha(R_i). \tag{29}$$

And, if I is an interval open on the left then, for any sequence $\{R_i\}_{i=1}^\infty$ of positive numbers with $R_i \geq R_0$, for all $i \in \mathbb{N}$, and $R_i \to \infty$ and for any sequence $\{J_n\}_{n=1}^\infty$ of compact intervals in I such that $\cup_{n=1}^\infty J_n = I$, we have that

$$\mathcal{U}_I^\alpha = \bigcap_{n=1}^\infty \bigcup_{i=1}^\infty \Pi_{J_n}^{-1}\mathcal{U}_{J_n}^\alpha(R_i). \tag{30}$$

The space $\mathcal{U}_I^\alpha(R)$ is closed in the topology inherited from $\mathcal{C}_{loc}(I,H_w)$ (and this implies that it is also closed in $\mathcal{C}_{loc}(I,H)$ since H is continuously included in H_w). Indeed, since the solutions in $\mathcal{U}_I^\alpha(R)$ are uniformly bounded by R in H, it suffices to show that $\mathcal{U}_I^\alpha(R)$ is closed in $\mathcal{C}_{loc}(I,B_H(R)_w)$. Since $\mathcal{C}_{loc}(I,B_H(R)_w)$ is metrizable, it suffices to work with sequences. Then, if $\{\mathbf{u}_n\}_n$ is a sequence in $\mathcal{U}_I^\alpha(R)$ which converges in $\mathcal{C}_{loc}(I,B_H(R)_w)$ to an element $\mathbf{u}$, then the *a priori* estimates (18), (19), (20) yield the compactness of this sequence in $\mathcal{C}_{loc}(I,H_w)$ and in $\mathcal{F}_I$. This compactness is sufficient to show that the limit function $\mathbf{u}$ is a solution of the Navier-Stokes-α model and, hence, belongs to $\mathcal{U}_I^\alpha(R)$, proving that this space is closed. Since this space is closed, there is no need to consider spaces analogous to $\mathcal{U}_I^\sharp(R)$ and $\mathcal{U}_I^\sharp$, as done for the Navier-Stokes equations.

Due to the representations (29) and (30), we see that $\mathcal{U}_I^\alpha$ is an $\mathcal{F}_\sigma$-set in $\mathcal{C}_{loc}(I,H_w)$, in the case I is closed and bounded on the left, and it is an $\mathcal{F}_{\sigma\delta}$-set, in the case I is open on the left. In any case, $\mathcal{U}_I^\alpha$ is a Borel set.

We now introduce an auxiliary functional space, $\mathcal{Y}_I$, which is directly connected with the *a priori* estimates for the solutions of the Navier-Stokes equations and of the Navier-Stokes-α model, with suitable compactness property. First, let J be a compact interval in $\mathbb{R}$, then we define

$$\begin{aligned}\mathcal{Y}_J(R) = \Bigg\{ \mathbf{u} \in \mathcal{C}_{loc}(J;H_w) : |\mathbf{u}(t)| \leq R, \|\mathbf{u}\|_{L^2(s,t;V)} \leq \frac{1}{\nu^{1/2}}R + \lambda_1^{1/4}\nu M(t-s)^{1/2}, \text{ and} \\ \|\partial_t \mathbf{u}\|_{L^2(s,t;D(A)')} \leq \frac{c}{\lambda_1^{1/4}\nu^{1/2}}R^2 + \frac{\nu^{3/2}}{\lambda_1^{3/4}}M + \nu^{5/2}\lambda_1^{1/4}M(t-s), \forall s,t \in J \Bigg\},\end{aligned} \tag{31}$$

where $c = \max\{c_1, c_2\}$ is a universal constant and $M = \max\{M_1, M_2\}$ is a non-dimensional constant which depends only on non-dimensional combinations of the terms ν, λ_1 and $\|\mathbf{f}\|_{L^\infty(I,H)}$. With these choices of

constants, for any $R \geq R_0$, if $|u_0| \leq R$, then the solutions of the Navier-Stokes equations and of the Navier-Stokes-α model with initial condition u_0 all satisfy the estimates in (31), for subsequent times, which is possible thanks to the a priori estimates (8), (9) and (10), and (18), (19), and (20). Thus, we have

$$\mathcal{U}_J(R),\, \mathcal{U}_J^\alpha(R) \subset \mathcal{Y}_J(R), \quad \forall R \geq R_0. \tag{32}$$

Now, let I be any interval in $\mathbb{R}$ and $\{J_n\}_{n=1}^\infty$ be a sequence of compact intervals in I such that $J_n \subset J_{n+1}$ and $\cup_{n=1}^\infty J_n = I$. Consider also a sequence $\{R_i\}_{i=1}^\infty$ of increasing real number such that $R_1 \geq R_0$ and $R_i \to \infty$ as $i \to \infty$. Define

$$\mathcal{Y}_I := \bigcap_{n=1}^\infty \bigcup_{i=1}^\infty \Pi_{J_n}^{-1} \mathcal{Y}_{J_n}(R_i). \tag{33}$$

Also, for a given $R \geq R_0$, we define

$$\mathcal{Y}_I(R) = \bigcap_{n=1}^\infty \Pi_{J_n}^{-1} \mathcal{Y}_{J_n}(R). \tag{34}$$

Observe that $\bigcup_{i=1}^\infty \mathcal{Y}_I(R_i) \subset \mathcal{Y}_I \subset \mathcal{C}_{loc}(I, H_w)$. The space $\mathcal{Y}_I(R)$ is independent of the choice of the intervals $\{J_n\}_n$, while the space $\mathcal{Y}_I$ is independent of the choice of both the intervals $\{J_n\}_n$ and the sequence $\{R_i\}_i$, although these properties are not really necessary.

Lemma 2.2. *Let J be a compact interval in $\mathbb{R}$. Then, $\mathcal{Y}_J(R)$ is a compact subset of $\mathcal{C}_{loc}(J, H_w)$.*

Proof. First, observe that since $\mathcal{Y}_J(R) \subset \mathcal{C}_{loc}(J, B_H(R)_w)$ then $\mathcal{Y}_J(R)$ is metrizable. Now, let $\{\mathbf{u}_n\}_n$ be a sequence in $\mathcal{Y}_J(R)$. It is clear that $\{\mathbf{u}_n\}$ is bounded in $L^2(J, V)$ and $\{\partial_t \mathbf{u}_n\}$ is bounded in $L^2(J, D(A)')$. Then, by Aubin's Compactness Theorem, we obtain that $\{\mathbf{u}_n\}$ is relatively compact in $L^2(J, H)$. Using all the information obtained before, we conclude that there exist a vector field $\mathbf{u}$ and a subsequence $\{\mathbf{u}_{n_k}\}$ such that

$$\begin{aligned} \mathbf{u}_{n_k} &\overset{*}{\rightharpoonup} \mathbf{u} \ \text{ in } L^\infty(J; H); \\ \mathbf{u}_{n_k} &\rightharpoonup \mathbf{u} \ \text{ in } L^2(J; V); \\ \partial_t \mathbf{u}_{n_k} &\rightharpoonup \partial_t \mathbf{u} \ \text{ in } L^2(J; D(A)'); \\ \mathbf{u}_{n_k} &\to \mathbf{u} \ \text{ in } L^2(J; H). \end{aligned}$$

Now, consider $\{\mathbf{w}_i\}_i$ a countable dense subset in $D(A)$ (the existence of such a set follows from the fact that $D(A)$ is separable). For each $i \in \mathbb{N}$, define the sequence $\{f_k^i\}_k$ such that, for each $k \in \mathbb{N}$, $f_k^i(t) = (\mathbf{u}_{n_k}(t), \mathbf{w}_i)$ for all $t \in J$. Note that $\{f_k^i\}_k$ is a sequence of continuous functions from J to $\mathbb{R}$, which is uniformly bounded and equicontinuous. Indeed, since

$$|f_k^i(t) - f_k^i(s)| = \left| \int_t^s (\partial_\tau \mathbf{u}_{n_k}(\tau), \mathbf{w}_i) \mathrm{d}\tau \right| \leq |t-s|^{1/2} \|\mathbf{w}_i\|_{D(A)} \|\partial_t \mathbf{u}_{n_k}\|_{L^2(t,s;D(A)')},$$

for all $t, s \in J$, then $\{f_k^i\}_k$ is equicontinuous. Thus, we can apply Arzelà-Ascoli Theorem to obtain that $\{f_k^i\}_k$ is relatively compact in $\mathcal{C}_{loc}(J, \mathbb{R})$, for each $i \in \mathbb{N}$. By Cantor's diagonal argument, we can construct a subsequence $\{f_{k_j}^i\}$ such that $f_{k_j}^i$ converges to $(\mathbf{u}(\cdot), \mathbf{w}^i)$ as $j \to \infty$, for all $i \in \mathbb{N}$. Now we use a triangulation argument to obtain the convergence of $\mathbf{u}_{n_{k_j}} \to \mathbf{u}$ in $\mathcal{C}_{loc}(J, H_w)$. More precisely, given $\varepsilon > 0$ and $\mathbf{v} \in H$, there exist $\mathbf{w} \in D(A)$ such that $|\mathbf{v} - \mathbf{w}| < \varepsilon/(6R)$, and $i_0 \in \mathbb{N}$ such that $\|\mathbf{w} - \mathbf{w}_{i_0}\|_{D(A)} < \lambda_1 \varepsilon/(6R)$. By the convergence of $\{f_{k_j}^i\}_j$ we conclude that there exists $N \in \mathbb{N}$ such that $\sup_{t \in J} |(\mathbf{u}_{n_{k_j}}(t) - \mathbf{u}(t), \mathbf{w}_i)| \leq \varepsilon/3$, for all $j \geq N$ and for all $i \in \mathbb{N}$.

Therefore, for all $j \geq N$, we have that

$$\begin{aligned}
&\sup_{t\in J} |(\mathbf{u}_{n_{k_j}}(t) - \mathbf{u}(t), \mathbf{v})| \leq \sup_{t\in J} |(\mathbf{u}_{n_{k_j}}(t) - \mathbf{u}(t), \mathbf{v} - \mathbf{w})| \\
&+ \sup_{t\in J} |(\mathbf{u}_{n_{k_j}}(t) - \mathbf{u}(t), \mathbf{w} - \mathbf{w}_{i_0})| + \sup_{t\in J} |(\mathbf{u}_{n_{k_j}}(t) - \mathbf{u}(t), \mathbf{w}_{i_0})| \\
&< \sup_{t\in J} |\mathbf{u}_{n_{k_j}}(t) - \mathbf{u}(t)|(|\mathbf{v} - \mathbf{w}| + |\mathbf{w} - \mathbf{w}_{i_0}|) + \frac{\varepsilon}{3} \\
&\leq 2R|\mathbf{v} - \mathbf{w}| + 2R\frac{1}{\lambda_1}\|\mathbf{w} - \mathbf{w}_{i_0}\|_{D(A)} + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

Finally, it is clear that $\mathbf{u}$ inherits the uniform estimates of the sequence in $\mathcal{Y}_J(R)$, so that $\mathbf{u}$ itself is in $\mathcal{Y}_J(R)$, completing the proof that $\mathcal{Y}_J(R)$ is compact. $\square$

Lemma 2.3. *Let I be any interval in $\mathbb{R}$ and $R > 0$. Then, the set $\mathcal{Y}_I(R)$ is compact in $\mathcal{C}_{loc}(I, H_w)$. Moreover, $\mathcal{Y}_I(R)$ is metrizable.*

Proof. As observed in the end of Section 2.4 in [25], compact subsets of $\mathcal{C}_{loc}(I, H_w)$ can be characterized as the sets K for which, for every compact interval $J \subset I$, the subset $\Pi_J K$ is equi-bounded with respect to the norm of H and equicontinuous with respect to the uniform structure of $\mathcal{C}_{loc}(J, H_w)$. Since $\Pi_J \mathcal{Y}_I(R) \subset \mathcal{Y}_J(R)$ and $\mathcal{Y}_J(R)$ is compact as proved in Lemma 2.2, these conditions are met, and we have that $\mathcal{Y}_I(R)$ is compact. Furthermore, since $\mathcal{Y}_I(R) \subset \mathcal{C}_{loc}(I, B_H(R)_w)$ then $\mathcal{Y}_I(R)$ is metrizable. $\square$

Lemma 2.4. *Let I be any interval in $\mathbb{R}$. Then, the space $\mathcal{Y}_I$, endowed with the topology inherited from $\mathcal{C}_{loc}(I, H_w)$, is a completely regular topological space. Moreover, $\mathcal{Y}_I$ contains the spaces $\mathcal{U}_I^\sharp$ and $\mathcal{U}_I^\alpha$.*

Proof. Since $\mathcal{Y}_I$ is a subspace of $\mathcal{C}_{loc}(I, H_w)$ then $\mathcal{Y}_I$ is a completely regular topological space. For the inclusions, suppose first that I is an interval closed and bounded on the left. Then, for every compact interval $J \subset I$ containing the left end point of I, and for all $R \geq R_0$, where R_0 is defined by (11), it follows from (32) that $\mathcal{U}_I^\sharp(R) \subset \Pi_J^{-1}\mathcal{Y}_J(R)$. Therefore, using (34) and (25) we conclude that $\mathcal{U}_I^\sharp \subset \mathcal{Y}_I$. Now, in order to prove that $\mathcal{U}_I^\alpha \subset \mathcal{Y}_I$, notice that from (32) we have that $\mathcal{U}_I^\alpha(R) \subset \Pi_J^{-1}\mathcal{Y}_I(R)$, for all $R \geq R_0$ and any compact interval $J \subset I$ containing the left end point of I. Thus, using (34) and (29) we conclude that $\mathcal{U}_I^\alpha \subset \mathcal{Y}_I$.

Now, suppose that I is open on the left. Again, we have from (32) that $\mathcal{U}_J^\sharp(R) \subset \mathcal{Y}_J(R)$, for all $R \geq R_0$ and any compact interval $J \subset I$. Thus, by (33) and (26), we conclude that $\mathcal{U}_I^\sharp \subset \mathcal{Y}_I$. Observe also that we have from (32) that $\mathcal{U}_J^\alpha(R) \subset \mathcal{Y}_J(R)$, for all $R \geq R_0$ and any compact interval $J \subset I$. Thus, (33) and (30) imply that $\mathcal{U}_I^\alpha \subset \mathcal{Y}_I$. $\square$

Remark 2.2. *If I is an interval closed and bounded on the left and $\{R_i\}_i$ is a sequence of positive numbers with $R_i \geq R_0$, for all $i \in \mathbb{N}$, and $R_i \to \infty$, then we have in fact showed in the proof of Lemma 2.4 that $\mathcal{U}_I^\alpha, \mathcal{U}_I^\sharp \subset \bigcup_{i=1}^\infty \mathcal{Y}_I(R_i)$. On the other hand, if I is an interval open on the left then $\mathcal{U}_I^\alpha, \mathcal{U}_I^\sharp \subset \mathcal{Y}_I$ but $\mathcal{U}_I^\alpha$ and $\mathcal{U}_I^\sharp$ might be not included in $\bigcup_{i=1}^\infty \mathcal{Y}_I(R_i)$.*

Lemma 2.5. *Let I be any interval in $\mathbb{R}$. Then, the space $\mathcal{Y}_I(R)$ contains the spaces $\mathcal{U}_I^\sharp(R)$ and $\mathcal{U}_I^\alpha(R)$, for all $R \geq R_0$, where R_0 is defined by (11).*

Proof. It is clear from estimates (8), (9) and (10) that, if $\mathbf{u} \in \mathcal{U}^\sharp(R)$, for some $R \geq R_0$, then $\mathbf{u} \in \Pi_J^{-1}\mathcal{Y}_I(R)$, for all compact interval $J \subset I$. Thus, it follows from (34) that $\mathcal{U}_I^\sharp(R) \subset \mathcal{Y}_I(R)$. With an analogous argument we can also prove that $\mathcal{U}_I^\alpha(R) \subset \mathcal{Y}_I(R)$, for all $R \geq R_0$. $\square$

Next, we prove an important convergence result concerning the trajectory spaces $\mathcal{U}_I^\sharp(R)$ and $\mathcal{U}_I^\alpha(R)$, based on Theorem 2.5.

Lemma 2.6. *Let $R \geq R_0$, where R_0 is defined by (11). Let $\mathcal{U}_I^\sharp(R)$ and $\mathcal{U}_I^\alpha(R)$ be given by (24) and (28), respectively. Then,*

$$\lim_{\alpha\to 0} dist_{\mathcal{Y}_I(R)}(\mathcal{U}_I^\alpha(R), \mathcal{U}_I^\sharp(R)) = 0$$

where

$$dist_{\mathcal{Y}_I(R)}(\mathcal{U}_I^\alpha(R), \mathcal{U}_I^\sharp(R)) = \sup_{\mathbf{w}\in\mathcal{U}_I^\alpha(R)} \inf_{\mathbf{u}\in\mathcal{U}_I^\sharp(R)} d(\mathbf{w},\mathbf{u}),$$

and d is any compatible metric in $\mathcal{Y}_I(R)$.

Proof. Suppose by contradiction that

$$\lim_{\alpha\to 0} \mathrm{dist}_{\mathcal{Y}_I(R)}(\mathcal{U}_I^\alpha(R), \mathcal{U}_I^\sharp(R)) \neq 0.$$

Thus, there exist $\varepsilon > 0$ and a sequence of $\{\alpha_n\}_n$, with $\alpha_n \to 0$ as $n \to \infty$, such that

$$\sup_{\mathbf{w}\in\mathcal{U}_I^{\alpha_n}(R)} \inf_{\mathbf{u}\in\mathcal{U}_I^\sharp(R)} d(\mathbf{w},\mathbf{u}) > \varepsilon, \ \forall n \in \mathbb{N}.$$

Observe that, from the definition of the supremum, we have that, given $r > 0$, there exists $\mathbf{w}_n \in \mathcal{U}_I^{\alpha_n}(R)$ such that

$$\inf_{\mathbf{u}\in\mathcal{U}_I^\sharp(R)} d(\mathbf{w}_n,\mathbf{u}) > \varepsilon - r, \ \forall n \in \mathbb{N}.$$

In particular, we can take $r = \varepsilon/2$ and obtain

$$\inf_{\mathbf{u}\in\mathcal{U}_I^\sharp(R)} d(\mathbf{w}_n,\mathbf{u}) > \frac{\varepsilon}{2}, \ \forall n \in \mathbb{N},$$

so that

$$d(\mathbf{w}_n,\mathbf{u}) > \frac{\varepsilon}{2}, \ \forall n \in \mathbb{N} \text{ and } \forall \mathbf{u} \in \mathcal{U}_I^\sharp(R). \tag{35}$$

On the other hand, we have that $|\mathbf{w}_n(t)|_H \leq R$ for all $t \in I$ and for all $n \in \mathbb{N}$. Thus, estimates (19) and (20) imply that $\{\mathbf{w}_n\}_n$ is bounded in $\mathcal{F}_I^b$ (see (3)). Therefore, there exist a subsequence $\{\mathbf{w}_{n_l}\}_l$ of $\{\mathbf{w}_n\}_n$ and a function $\mathbf{u} \in \mathcal{F}_I^b$ such that $\mathbf{w}_{n_l} \to \mathbf{u}$ with respect to τ. Using Theorem 2.5, we conclude that $\mathbf{u} \in \mathcal{U}_I^\sharp(R)$. Moreover, for any compact interval $J \subset I$, we have that $\{\mathbf{w}_n\}_n$ is in $\mathcal{Y}_J(R)$. Then using Lemma 2.2 we conclude that there exists a subsequence of $\{\mathbf{w}_n\}_n$ that converges in the topology of weak converge in H uniformly on J. We can now use Cantor's diagonal argument to obtain a subsequence $\{\mathbf{w}_{n_k}\}_{n_k}$ that converges to $\mathbf{u}$ in the topology of weak converge in H uniformly on J, for any compact interval $J \subset I$. Observe that this lead us to a contradiction with (35) since d is a compatible metric with the topology of $\mathcal{Y}_I(R)$. □

2.6 Statistical solutions

The notion of statistical solutions that is considered here was introduced by Foias, Rosa and Temam, in [23, 24] (see also [25, 26]). We recall this definition in the context of the Navier-Stokes equations and introduce a corresponding definition for the Navier-Stokes-α model.

2.6.1 Time-dependent statistical solutions

We start with the definition of Vishik-Fursikov measure for the Navier-Stokes equations which will give rise to the statistical solutions for the Navier-Stokes equations.

Definition 2.4. *Let $I \subset \mathbb{R}$ be an interval. We say that a Borel probability measure ρ in $\mathcal{C}_{loc}(I, H_w)$ is a **Vishik-Fursikov measure** over I if ρ satisfies the following*

(i) ρ is carried by $\mathcal{U}_I^\sharp$;

(ii) $t \mapsto \int_{\mathcal{U}_I^\sharp} |\mathbf{u}(t)|^2 d\rho(\mathbf{u}) \in L^\infty_{loc}(I)$;

(iii) if I is closed and bounded on the left, with left end point t_0, then for all $\psi \in \Psi$ we have that

$$\lim_{t \to t_0^+} \int_{\mathcal{U}_I^\sharp} \psi(|\mathbf{u}(t)|^2) d\rho(\mathbf{u}) = \int_{\mathcal{U}_I^\sharp} \psi(|\mathbf{u}(t_0)|^2) d\rho(\mathbf{u}),$$

where $\Psi := \{\psi \in \mathcal{C}^1([0,\infty)) : \psi \geq 0, \psi' \geq 0$ and $\sup_{t\geq 0} \psi'(t) < \infty\}$.

Observe that in this definition we only require the measure ρ to be carried by $\mathcal{U}_I^\sharp$. But we really want to have ρ carried by $\mathcal{U}_I$, which is the trajectory space of Leray-Hopf weak solutions. And this is what in fact happens. More precisely, in [25, Theorem 4.1], it was proved that for an arbitrary interval $I \subset \mathbb{R}$, any Vishik-Fursikov measure over I is carried by $\mathcal{U}_I$.

Next we present the definition of a Foias-Prodi statistical solution, which is a family of measures on the phase space, satisfying a Liouville-type equation and some regularity properties. Let us denote by $\mathbb{F}$ the operator defined on V as $\mathbb{F}(\mathbf{u}) = \mathbf{f} - \nu A\mathbf{u} - B(\mathbf{u}, \mathbf{u})$, with values in V'. A function $\Phi : H \to \mathbb{R}$ is called a **cylindric test function** if

$$\Phi(\mathbf{u}) = \varphi((\mathbf{u}, \mathbf{v}_1), \ldots, (\mathbf{u}, \mathbf{v}_k)),$$

where $k \in \mathbb{N}$, φ is a continuously differentiable real-valued function on $\mathbb{R}^k$ with compact support, and $\mathbf{v}_1, \ldots, \mathbf{v}_k$ belong to V. For such Φ, we denote by Φ' its Fréchet derivative in H, which has the form

$$\Phi'(\mathbf{u}) = \sum_{j=1}^{k} \partial_j \varphi((\mathbf{u}, \mathbf{v}_1), \ldots, (\mathbf{u}, \mathbf{v}_k))\mathbf{v}_j,$$

where $\partial_j \varphi$ is the derivative of φ with respect to its j-th coordinate.

Definition 2.5. *A family $\{\mu_t\}_{t\geq 0}$ of Borel probabilities on H is a **statistical solution** of the 3D Navier-Stokes equations if it satisfies:*

(i) the Liouville type equation

$$\frac{d}{dt} \int_H \Phi(\mathbf{u})\, d\mu_t(\mathbf{u}) = \int_H \langle \mathbb{F}(\mathbf{u}), \Phi'(\mathbf{u}) \rangle\, d\mu_t(\mathbf{u}),$$

in the distributional sense in $t \geq 0$, for all cylindric test functions Φ;

(ii) the function

$$t \mapsto \int_H \phi(\mathbf{u})\, d\mu_t(\mathbf{u})$$

is measurable in $t \geq 0$ for all continuous functional $\phi : H \to \mathbb{R}$;

(iii) the function

$$t \mapsto \int_H |\mathbf{u}|_H^2 \, d\mu_t(\mathbf{u})$$

belongs to $L^\infty_{loc}(0,\infty)$*;*

(iv) the function

$$t \mapsto \int_H \|\mathbf{u}\|_V^2 \, d\mu_t(\mathbf{u})$$

belongs to $L^1_{loc}(0,\infty)$*;*

(v) the mean strengthened energy inequality holds, i.e.,

$$\frac{1}{2}\frac{d}{dt}\int_H \psi\left(|\mathbf{u}|_H^2\right) \, d\mu_t(\mathbf{u}) + \nu \int_H \psi'\left(|\mathbf{u}|_H^2\right) \|\mathbf{u}\|_V^2 \, d\mu_t(\mathbf{u}) \leq \int_H \psi'\left(|\mathbf{u}|_H^2\right) \langle \mathbf{f}, \mathbf{u}\rangle \, d\mu_t(\mathbf{u})$$

in the distributional sense in $t \geq 0$*, for all* $\psi \in \Psi$*;*

(vi) and the function

$$t \mapsto \int_H \psi(|\mathbf{u}|_H^2) \, d\mu_t(\mathbf{u})$$

is continuous at $t = 0$*, for all* $\psi \in \Psi$*.*

In [25], it was proved that for any Borel probability measure μ_0 on H such that $\int_H |u|^2 d\mu_0(u) < \infty$, there exists a Vishik-Fursikov measure ρ over $I = [t_0, \infty)$ such that $\Pi_{t_0}\rho = \mu_0$. Furthermore, if ρ is a Vishik-Fursikov measure then $\mu_t := \Pi_t \rho$, $t \in I$, is a statistical solution in the sense of Definition 2.5. This yields a particular type of statistical solution:

Definition 2.6. *Let $I \subset \mathbb{R}$ be an arbitrary interval. A* ***Vishik-Fursikov statistical solution*** *of the Navier-Stokes equations over I is a statistical solution $\{\rho_t\}_{t\in I}$ such that $\rho_t = \Pi_t \rho$, for all $t \in I$, for some Vishik-Fursikov measure ρ over the interval I.*

Inspired by the definition of a Vishik-Fursikov measure, we define the α-Vishik-Fursikov measure which will give rise to the statistical solutions for the Navier-Stokes-α model.

Definition 2.7. *Let $I \subset \mathbb{R}$ be an interval and $\alpha > 0$. We say that a Borel probability measure ρ_α in $\mathcal{C}_{loc}(I, H)$ is an* α***-Vishik-Fursikov measure*** *over I if ρ_α satisfies the following*

(i) ρ_α is carried by $\mathcal{U}_I^\alpha$;

(ii) $t \mapsto \int_{\mathcal{U}_I^\alpha} |\mathbf{w}(t)|^2 d\rho_\alpha(\mathbf{w}) \in L^\infty_{loc}(I)$.

Since we chose to work with solutions of the Navier-Stokes-α model in the sense of Definition 2.3, and thanks to condition (iii) of that definition, it is natural to define α-Vishik-Fursikov measures on $\mathcal{C}_{loc}(I, H)$. Since $\mathcal{U}_I^\alpha$ is contained in $\mathcal{C}_{loc}(I, H)$ as a set, the condition (ii) of the Definition 2.7 above makes sense.

As another remark, since $\mathcal{C}_{loc}(I, H)$ is continuously included in $\mathcal{C}_{loc}(I, H_w)$ and the space $\mathcal{U}_I^\alpha(R)$ is closed in $\mathcal{C}_{loc}(I, H_w)$ (see Section 2.5), there is no need to consider a space analogous to $\mathcal{U}_I^\sharp$, as in Definition 2.4.

Remark 2.3. *Observe that $\mathcal{C}_{loc}(I, H)$ is a Polish space (since it is Fréchet and separable) so that every Borel probability measure ρ is tight. Moreover, since the Borel sets of H are the same as the Borel sets of H_w (see [21]) then the Borel sets of $\mathcal{C}_{loc}(I, H)$ are the same as the Borel sets of $\mathcal{C}_{loc}(I, H_w)$. Therefore, every Borel probability measure in $\mathcal{C}_{loc}(I, H_w)$ is tight.*

We also define an **α-Vishik-Fursikov statistical solution** of the Navier-Stokes-α model over an arbitrary interval I as a family $\{\rho_t^\alpha\}_{t\in I}$ of Borel probability measures on H such that $\rho_t^\alpha = \Pi_t\rho_\alpha$, for all $t \in I$, for some α-Vishik-Fursikov measure ρ_α over the interval I.

The existence of α-Vishik-Fursikov measures is easy to obtain. For instance, any Dirac measure in the trajectory space $\mathcal{U}_I^\alpha$ is an α-Vishik-Fursikov measure over I. Furthermore, given an initial Borel probability measure μ_0 in H with finite energy we can construct an α-Vishik-Fursikov measure ρ_α over $[0,\infty)$ such that $\Pi_0\rho_\alpha = \mu_0$. Indeed, since the Navier-Stokes-α model is well-posed, the solution semigroup $\{S_\alpha(t)\}_{t\geq 0}$ is well-defined. Moreover, the operator $S_\alpha(\cdot) : H \to \mathcal{C}_{loc}(I, H)$, defined as $S_\alpha(\cdot)\mathbf{w}_0 = \mathbf{w}(\cdot)$ for $\mathbf{w}_0 \in H$, where $S_\alpha(t)\mathbf{w}_0 = \mathbf{w}(t)$ for all $t \in I$, is continuous. Therefore, given an initial Borel probability measure μ_0 on H, we may define the Borel probability measure ρ_α as

$$\rho_\alpha(E) = \mu_0(S_\alpha(\cdot)^{-1}E), \qquad \text{for all } E \in \mathcal{C}_{loc}(I, H) \text{ Borel, where } I = [0,\infty).$$

By construction, it is clear that ρ_α is carried by the set $\mathcal{U}_I^\alpha$. Moreover, since for each $\mathbf{w} \in \mathcal{U}_I^\alpha$ it holds that $|\mathbf{w}(t)| \leq |\mathbf{w}(0)| + 1/(\lambda_1^2\nu_1^2)\|\mathbf{f}\|^2_{L^\infty(I,H)}$, for all $t \geq 0$, and $\rho_\alpha = S_\alpha(\cdot)\mu_0$, then, using the Change of Variables Theorem (see Section 2.2), we obtain that

$$\begin{aligned}\int_{\mathcal{U}_I^\alpha} |\mathbf{w}(t)|^2 d\rho_\alpha(\mathbf{w}) &\leq \int_{\mathcal{U}_I^\alpha} |\mathbf{w}(0)|^2 d\rho_\alpha(\mathbf{w}) + \frac{1}{\lambda_1^2\nu_1^2}\|\mathbf{f}\|^2_{L^\infty(I,H)} \\ &= \int_H |(S_\alpha(\cdot)\mathbf{u})(0)|^2 d\mu_0(\mathbf{u}) + \frac{1}{\lambda_1^2\nu_1^2}\|\mathbf{f}\|^2_{L^\infty(I,H)} \\ &= \int_H |\mathbf{u}|^2 d\mu_0(\mathbf{u}) + \frac{1}{\lambda_1^2\nu_1^2}\|\mathbf{f}\|^2_{L^\infty(I,H)}.\end{aligned}$$

Consequently, since μ_0 has finite energy we obtain that $t \mapsto \int_{\mathcal{U}_I^\alpha} |\mathbf{w}(t)|^2 d\rho_\alpha(\mathbf{w}) \in L^\infty_{loc}([0,\infty))$. Therefore, ρ_α is an α-Vishik-Fursikov measure, and it is straightforward that $\Pi_0\rho_\alpha = \mu_0$.

Nevertheless, since the well-posedness for the 3D Navier-Stokes equations is an open problem, the abstract definition of Vishik-Fursikov measure is essential in the context of this article.

2.6.2 Stationary statistical solutions

As already mentioned in the Introduction, stationary statistical solutions are valuable in the study of turbulence in statistical equilibrium in time, yielding, in particular, rigorous proofs of important statistical estimates. The concept of stationary statistical solution represents a generalization of invariant measures for the semigroup generated by an equation. For instance, in 2D, since the Navier-Stokes equations has a well-defined semigroup, stationary statistical solutions are, under certain hypothesis, precisely the invariant measures for the semigroup; see [21] for more details.

Suppose I is an interval unbounded on the right, hence having one of the following forms: $I = [t_0,\infty)$, $I = (t_0,\infty)$ or $I = \mathbb{R}$. We introduce the time-shift operator σ_τ defined for any $\tau > 0$ by

$$\begin{aligned}\sigma_\tau : \mathcal{C}_{loc}(I, H_w) &\to C_{loc}(I, H_w) \\ \mathbf{u} &\mapsto (\sigma_\tau\mathbf{u})(t) = \mathbf{u}(t+\tau),\ \forall t \in I.\end{aligned}$$

An **invariant Vishik-Fursikov measure** over I is a Vishik-Fursikov measure ρ which is invariant with respect to the translation semigroup $\{\sigma_\tau\}_{\tau\geq 0}$, in the sense that $\sigma_\tau\rho = \rho$ for all $\tau \geq 0$, i.e., $\rho(E) = \rho(\sigma_\tau^{-1}E)$, for all Borel set E in $\mathcal{C}_{loc}(I, H_w)$.

The family of projections $\{\Pi_t\rho\}_{t\in I}$ of an invariant Vishik-Fursikov measure ρ has the property that any statistical information

$$\int_H \phi(\mathbf{u})\mathrm{d}(\Pi_t\rho)(\mathbf{u}) = \int_H \phi(\mathbf{u}(t))\mathrm{d}\rho(\mathbf{u})$$

is independent of the time variable $t \in I$, for any $\phi \in \mathcal{C}_b(H_w)$. In fact, the measure $\Pi_t\rho$ itself is independent of t and is a statistical solution in the sense of Foias-Prodi which is time-independent (what is called a stationary statistical solution in the sense of Foias-Prodi). This yields a particular type of stationary statistical solution as stated precisely below.

Definition 2.8. *A **stationary Vishik-Fursikov statistical solution** on H is a Borel probability measure ρ_0 on H which is a projection $\rho_0 = \Pi_t\rho$, at an arbitrary time $t \in I$, of an invariant Vishik-Fursikov measure ρ over an interval I unbounded on the right.*

3 Convergence of statistical solutions of the α-model as α vanishes

In this section we present the main results of this paper. We prove that, under certain conditions, statistical solutions of the Navier-Stokes-α model converge to statistical solutions of the 3D Navier-Stokes equations as $\alpha \to 0$. We first prove this result for time-dependent statistical solutions and then we address the particular case of stationary statistical solutions.

As defined in Section 2.2, we mean by $\rho_\alpha \overset{*}{\rightharpoonup} \rho$ in $\mathcal{P}(X)$ that

$$\lim_{\alpha\to 0} \int_X \varphi(x)\mathrm{d}\rho_\alpha(x) = \int_X \varphi(x)\mathrm{d}\rho(x), \text{ for all } \varphi \in \mathcal{C}_b(X).$$

Observe that, by Theorem 2.2, in the case when X is a completely regular Hausdorff space, which is the case for us, this is exactly the weak convergence $\overset{w}{\to}$ discussed in Section 2.2.

3.1 Time-dependent statistical solutions

In this section we state some results concerning the convergence of α-Vishik-Fursikov measures and statistical solutions as α vanishes. We will see that it suffices to impose some condition of uniform boundedness on the mean kinetic energy over the α-Vishik-Fursikov measures (or, equivalently, on the α-Vishik-Fursikov statistical solutions). This will assure the tightness of the α-Vishik-Fursikov measures and yield a convergent subsequence. The convergence of these measures will imply the convergence of the associated statistical solutions in a suitable sense, described as follows.

Let $I \subset \mathbb{R}$ be an arbitrary interval. We say that a family $\{\{\mu_t^\alpha\}_{t\in I}\}_{\alpha>0}$ of α-Vishik-Fursikov statistical solutions over I converges, as $\alpha \to 0$, to a Vishik-Fursikov statistical solution over I if there exists a Vishik-Fursikov statistical solution $\{\mu_t\}_{t\in I}$ over I such that

$$\lim_{\alpha\to 0} \int_H \phi(\mathbf{u})d\mu_t^\alpha(\mathbf{u}) = \int_H \phi(\mathbf{u})d\mu_t(\mathbf{u}), \quad \forall t \in I \text{ and } \forall \phi \in \mathcal{C}_b(H_w). \tag{36}$$

Recall that a Vishik-Fursikov statistical solution (α-Vishik-Fursikov statistical solution) over an interval I is a family of Borel probability measures $\{\mu_t\}_{t\in I}$ given by $\mu_t = \Pi_t\rho$, for all $t \in I$, where ρ is some Vishik-Fursikov measure (α-Vishik-Fursikov measure) over I.

Thus, if we have a family of α-Vishik-Fursikov measures $\{\rho_\alpha\}_{\alpha>0}$ that converges to a Vishik-Fursikov measure ρ in $\mathcal{P}(\mathcal{Y}_I)$, i.e.,

$$\lim_{\alpha\to 0} \int_{\mathcal{Y}_I} \varphi(\mathbf{u})d\rho_\alpha(\mathbf{u}) = \int_{\mathcal{Y}_I} \varphi(\mathbf{u})d\rho(\mathbf{u}), \quad \forall \varphi \in \mathcal{C}_b(\mathcal{Y}_I), \tag{37}$$

then, as a simple consequence of the Change of Variables Theorem (see Section 2.2), we obtain the convergence of the α-Vishik-Fursikov statistical solutions $\{\Pi_t\rho_\alpha\}_{t\in I}$ to the Vishik-Fursikov statistical solution $\{\Pi_t\rho\}_{t\in I}$.

The measures ρ and ρ_α are defined on $\mathcal{C}_{loc}(I, H_w)$ and $\mathcal{C}_{loc}(I, H)$, respectively. Since $\mathcal{C}_{loc}(I, H)$ is included in $\mathcal{C}_{loc}(I, H_w)$ we may consider them as measures on $\mathcal{C}_{loc}(I, H_w)$. In fact, since they are actually carried by the

space of solutions $\mathcal{U}_I$ and $\mathcal{U}_I^\alpha$, respectively, and these spaces are included in $\mathcal{Y}_I$, we may consider them restricted to $\mathcal{Y}_I$. With this in mind and for the sake of simplicity we consider, in what follows, the measures ρ and ρ_α as measures on $\mathcal{Y}_I$. This is the same reason why we used $\mathcal{Y}_I$ in the convergence (37).

We state first a lemma which is essentially Theorem 2.2 translated into the framework of interest in this section.

Lemma 3.1. *Let I be an arbitrary interval in $\mathbb{R}$ and $\{\rho_\alpha\}_{\alpha>0}$ be a family in $\mathcal{P}(\mathcal{Y}_I, t)$. Suppose that for every $\varepsilon > 0$ there exists $R > 0$ such that $\rho_\alpha(\mathcal{Y}_I \setminus \mathcal{Y}_I(R)) < \varepsilon$, for all $\alpha > 0$. Then, there exists a sequence $\{\rho_{\alpha_n}\}_n \subset \{\rho_\alpha\}_{\alpha>0}$, with $\alpha_n \to 0$ as $n \to \infty$, such that $\rho_{\alpha_n} \overset{*}{\rightharpoonup} \rho$ in $\mathcal{P}(\mathcal{Y}_I)$ as $n \to \infty$.*

Proof. First, observe that since $\{\alpha \in \mathbb{R} : \alpha > 0\}$ is a totally ordered set we can subtract a sequence $\{\rho_{\alpha_n}\}_n$ from $\{\rho_\alpha\}_\alpha$, with $\alpha_n \to 0$ as $n \to \infty$. Moreover, observe that $\mathcal{Y}_I$ is a completely regular Hausdorff space and $\mathcal{Y}_I(R)$ is compact, for any $R > 0$. Therefore, it is clear that the sequence $\{\rho_{\alpha_n}\}_{n\in\mathbb{N}}$ fulfills the hypothesis of Theorem 2.2. Therefore, there exist $\rho \in \mathcal{P}(\mathcal{Y}_I, t)$ and a subsequence, which we still denote by $\{\rho_{\alpha_n}\}_n \subset \{\rho_\alpha\}_{\alpha>0}$, such that $\rho_{\alpha_n} \overset{*}{\rightharpoonup} \rho$ in $\mathcal{P}(\mathcal{Y}_I)$. $\square$

We would like to highlight that the use of a sequence in Lemma 3.1 is only for the sake of simplicity. Notice that Theorem 2.2 can be applied to the net $\{\rho_\alpha\}_\alpha$, as in Lemma 3.1, and it yields the existence of a convergent subnet of $\{\rho_\alpha\}_\alpha$. Then, all the subsequent results are also valid if we work with a convergent subnet instead of a sequence.

Observe that in order to apply the last lemma to a family of α-Vishik Fursikov measures we need the sequence to be uniformly tight. A natural condition that yields this uniform tightness is obtained by imposing a uniform boundedness condition on the mean kinetic energy of this family. This uniform boundedness of the mean kinetic energy will also be important to yield that the limit measure has finite mean kinetic energy and is a Vishik-Fursikov measure. This uniform boundedness can be imposed in different ways. We start with the following:

Proposition 3.1. *Let I be any interval in $\mathbb{R}$. Let $\{\rho_\alpha\}_{\alpha>0}$ be a family of Borel probability measures on $\mathcal{Y}_I$ such that, for each $\alpha > 0$, ρ_α is an α-Vishik-Fursikov measure over I and $\sup_{t\in I} \int_{\mathcal{Y}_I} |\mathbf{w}(t)|^2 d\rho_\alpha(\mathbf{w}) \leq C$, for all $\alpha > 0$, for some constant $C \geq 0$. Then, there exists a sequence $\{\rho_{\alpha_n}\}_n \subset \{\rho_\alpha\}_{\alpha>0}$, with $\alpha_n \to 0$, converging to a Borel probability measure ρ in $\mathcal{Y}_I$. Moreover, ρ is carried by $\mathcal{U}_I^\sharp$ and $\Pi_{\mathring{I}}\rho$ is a Vishik-Fursikov measure over $\mathring{I}$.*

Proof. In order to prove the convergence we will use Lemma 3.1. First, recall that every α-Vishik-Fursikov measure is tight (Remark 2.3). Next, we check the uniform tightness condition. Observe that it is enough to prove that for all $\varepsilon > 0$ there exists $R \geq R_0$ such that $\rho_\alpha(\mathcal{U}_I^\alpha \setminus \mathcal{U}_I^\alpha(R)) < \varepsilon$ for all $\alpha > 0$. Indeed, since ρ_α is carried by $\mathcal{U}_I^\alpha$ we have that

$$\begin{aligned} \rho_\alpha(\mathcal{Y}_I \setminus \mathcal{Y}_I(R)) &= \rho_\alpha(\mathcal{Y}_I \cap (\mathcal{Y}_I(R))^c) = \rho_\alpha(\mathcal{U}_I^\alpha \cap (\mathcal{Y}_I(R))^c) \\ &\leq \rho_\alpha(\mathcal{U}_I^\alpha \cap (\mathcal{U}_I^\alpha(R))^c) = \rho_\alpha(\mathcal{U}_I^\alpha \setminus \mathcal{U}_I^\alpha(R)), \end{aligned}$$

where the inequality holds since $\mathcal{U}_I^\alpha(R) \subset \mathcal{Y}_I(R)$, which implies that $\mathcal{Y}_I(R)^c \subset (\mathcal{U}_I^\alpha(R))^c$.

Let $R \geq R_0$. If $\mathcal{U}_I^\alpha \setminus \mathcal{U}_I^\alpha(R)$ is not empty, then for all $\mathbf{w} \in \mathcal{U}_I^\alpha \setminus \mathcal{U}_I^\alpha(R)$, there exists $t_{\mathbf{w}} \in I$ such that $|\mathbf{w}(t_{\mathbf{w}})|^2 > R$. Thus, $\sup_{t\in I} |\mathbf{w}(t)|^2 \geq R$ and

$$R\rho_\alpha(\mathcal{U}_I^\alpha \setminus \mathcal{U}_I^\alpha(R)) \leq \int_{\mathcal{U}_I^\alpha \setminus \mathcal{U}_I^\alpha(R)} \sup_{t\in I} |\mathbf{w}(t)|^2 d\rho_\alpha(\mathbf{w}). \tag{38}$$

Since for all $\mathbf{w} \in \mathcal{U}_I^\alpha$ we have that

$$|\mathbf{w}(t)|^2 \leq |\mathbf{w}(t')|^2 e^{-\lambda_1\nu(t-t')} + \frac{1}{\lambda_1^2\nu^2}\|\mathbf{f}\|_{L^\infty(t',t;H)}^2 \left(1 - e^{-\lambda_1\nu(t-t')}\right),$$

for all $t, t' \in I$ with $t > t'$, then

$$\int_{\mathcal{U}_I^\alpha} \sup_{t>t'} |\mathbf{w}(t)|^2 d\rho_\alpha \leq \int_{\mathcal{U}_I^\alpha} |\mathbf{w}(t')|^2 d\rho_\alpha + \frac{1}{\lambda_1^2 \nu^2} \|\mathbf{f}\|^2_{L^\infty(I,H)}.$$

Therefore, by the last inequality and the hypothesis $\sup_{t\in I} \int_{\mathcal{Y}_I} |\mathbf{w}(t)|^2 d\rho_\alpha(\mathbf{w}) \leq C$, for all $\alpha > 0$, it follows that

$$\int_{\mathcal{U}_I^\alpha} \sup_{t>t'} |\mathbf{w}(t)|^2 d\rho_\alpha \leq C_1,$$

for all $t' \in I$ and for all $\alpha > 0$, where $C_1 = C + 1/(\lambda_1^2\nu^2)\|\mathbf{f}\|^2_{L^\infty(I,H)}$.

Also observe that, for all $\mathbf{w} \in \mathcal{U}_I^\alpha$, $\mathbf{w} \in \mathcal{C}_{loc}(I, H)$ so that the function defined by $f(t') = \sup_{t>t'} |\mathbf{w}(t)|^2$ for $t' \in \bar{I}$, is continuous. Thus, for any $t_0 \in \bar{I}$ and any sequence $\{t'_k\}_k \subset I$ such that $t'_k \to t_0^+$, using the Monotone Convergence Theorem, we find that

$$\int_{\mathcal{U}_I^\alpha} \sup_{t>t_0} |\mathbf{w}(t)|^2 d\rho_\alpha \leq \int_{\mathcal{U}_I^\alpha} |\mathbf{w}(t')|^2 d\rho_\alpha \leq C_1, \quad \forall t_0 \in \bar{I} \text{ and } \forall \alpha > 0.$$

In particular, we obtain

$$\int_{\mathcal{U}_I^\alpha} \sup_{t\in I} |\mathbf{w}(t)|^2 d\rho_\alpha \leq C_1, \quad \forall \alpha > 0.$$

We use the last estimate in (38) to obtain, in the case that $\mathcal{U}_I^\alpha \setminus \mathcal{U}_I^\alpha(R)$ is not empty, that

$$\rho_\alpha(\mathcal{U}_I^\alpha \setminus \mathcal{U}_I^\alpha(R)) \leq \frac{C_1}{R}, \qquad \forall \alpha > 0.$$

If $\mathcal{U}_I^\alpha \setminus \mathcal{U}_I^\alpha(R)$ is empty, then this estimate is trivially valid. Then, given $\varepsilon > 0$, take $R = \max\{2C_1/\varepsilon, R_0\}$, so that

$$\rho_\alpha(\mathcal{U}_I^\alpha \setminus \mathcal{U}_I^\alpha(R)) < \varepsilon, \qquad \forall \alpha > 0.$$

This shows that $\{\rho_\alpha\}_{\alpha>0}$ is uniformly tight.

Then, we can apply Lemma 3.1 and obtain the existence of a measure $\rho \in \mathcal{P}(\mathcal{Y}_I, t)$ and a sequence $\{\rho_{\alpha_n}\}_n \subset \{\rho_\alpha\}_{\alpha>0}$, with $\alpha_n \to 0$ as $n \to \infty$, such that $\rho_{\alpha_n} \overset{*}{\rightharpoonup} \rho$ in $\mathcal{P}(\mathcal{Y}_I)$ as $n \to \infty$.

Next, we prove that ρ is carried by $\mathcal{U}_I^\sharp$. In order to do so, we define, for each $\varepsilon > 0$ and $R \geq R_0$, the set

$$\mathcal{Y}_\varepsilon(R) = \{u \in \mathcal{Y}_I(R) : \mathrm{dist}_{\mathcal{Y}_I(R)}(u, \mathcal{U}_I^\sharp(R)) < \varepsilon\}.$$

Observe that $\mathcal{Y}_I(R) \setminus \mathcal{Y}_\varepsilon(R)$ and $\overline{\mathcal{Y}_{\varepsilon/2}(R)}$ are disjoint closed sets in $\mathcal{Y}_I(R)$. Since $\mathcal{Y}_I(R)$ is a compact Hausdorff space then, by Urysohn's Lemma (see e.g. [1]), for each $\varepsilon > 0$, there exists a continuous function $\varphi_\varepsilon^R : \mathcal{Y}_I(R) \to [0, 1]$, such that $\varphi_\varepsilon^R(\mathbf{u}) = 1$ for all $\mathbf{u} \in \overline{\mathcal{Y}_{\varepsilon/2}(R)}$ and $\varphi_\varepsilon^R(\mathbf{u}) = 0$ for all $\mathbf{u} \in \mathcal{Y}_I(R) \setminus \mathcal{Y}_\varepsilon(R)$. Now, we define an extension of φ_ϵ^R to $\mathcal{Y}_I$, $\tilde{\varphi}_\varepsilon^R : \mathcal{Y}_I \to [0, 1]$ as $\tilde{\varphi}_\varepsilon^R(\mathbf{u}) = \varphi_\varepsilon^R(\mathbf{u})$, for all $\mathbf{u} \in \mathcal{Y}_I(R)$ and $\tilde{\varphi}_\varepsilon^R(\mathbf{u}) = 0$, for all $\mathbf{u} \in \mathcal{Y}_I \setminus \mathcal{Y}_I(R)$. Since $\mathcal{Y}_I(R)$ is closed, it is easy to see that $\tilde{\varphi}_\varepsilon^R$ is upper semicontinuous. Then, using that $\rho_{\alpha_n} \overset{*}{\rightharpoonup} \rho$ in $\mathcal{P}(\mathcal{Y}_I)$ and Lemma 2.1, we obtain that

$$\int_{\mathcal{Y}_I} \tilde{\varphi}_\varepsilon^R(\mathbf{u}) d\rho(\mathbf{u}) \geq \limsup_{n\to\infty} \int_{\mathcal{Y}_I} \tilde{\varphi}_\varepsilon^R(\mathbf{u}) d\rho_{\alpha_n}(\mathbf{u}) = \limsup_{n\to\infty} \int_{\mathcal{U}_I^{\alpha_n}(R)} \varphi_\varepsilon^R(\mathbf{u}) d\rho_{\alpha_n}(\mathbf{u}). \tag{39}$$

Since we have, by Lemma 2.6, that $\lim_{n\to\infty} \mathrm{dist}_{\mathcal{Y}_I(R)}(\mathcal{U}_I^{\alpha_n}(R), \mathcal{U}_I^\sharp(R)) = 0$ then, given $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that, for each $m \geq N_0$,

$$\mathrm{dist}_{\mathcal{Y}_I(R)}(\mathcal{U}_I^{\alpha_m}(R), \mathcal{U}_I^\sharp(R)) < \varepsilon/2,$$

which implies that for all $\mathbf{u} \in \mathcal{U}_I^{\alpha_m}(R)$, $m \geq N_0$, $\mathrm{dist}_{\mathcal{Y}_I(R)}(\mathbf{u}, \mathcal{U}_I^\sharp(R)) < \varepsilon/2$. In other words, given $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $\mathcal{U}_I^{\alpha_m}(R) \subset \mathcal{Y}_{\varepsilon/2}(R)$, for all $m \geq N_0$ and therefore $\varphi_\varepsilon^R(\mathbf{u}) = 1$ for all $\mathbf{u} \in \mathcal{U}_I^{\alpha_m}(R)$, for all $m \geq N_0$. Thus, for all $m \geq N_0$,

$$\int_{\mathcal{U}_I^{\alpha_m}(R)} \varphi_\varepsilon^R(\mathbf{u}) d\rho^{\alpha_m}(\mathbf{u}) = \rho_{\alpha_m}(\mathcal{U}_I^{\alpha_m}(R)) \geq 1 - \frac{C_1}{R},$$

hence, using the last estimate in (39), we conclude that

$$\int_{\mathcal{Y}_I} \tilde{\varphi}_\varepsilon^R(\mathbf{u}) d\rho(\mathbf{u}) \geq 1 - \frac{C_1}{R}.$$

Therefore,

$$\rho(\mathcal{Y}_\varepsilon(R)) \geq \int_{\mathcal{Y}_I} \tilde{\varphi}_\varepsilon^R(\mathbf{u}) d\rho(\mathbf{u}) \geq 1 - \frac{C_1}{R}.$$

Since $\mathcal{U}_I^\sharp(R) = \cap_{j=1}^\infty \mathcal{Y}_{\varepsilon_j}(R)$ for any sequence of positive numbers $\varepsilon_j \to 0$, we obtain that

$$\rho(\mathcal{U}_I^\sharp(R)) \geq 1 - C_1/R, \quad \forall R \geq R_0.$$

Now, in order to prove that $\rho(\mathcal{U}_I^\sharp) = 1$, we first suppose that I is closed and bounded on the left. In this case we can write $\mathcal{U}_I^\sharp = \cup_{i=1}^\infty \mathcal{U}_I^\sharp(R_i)$, for any sequence $\{R_i\}_i$ with $R_0 \leq R_i \leq R_{i+1}$, for all $i \in \mathbb{N}$, and $R_i \to \infty$. Observe that $\mathcal{U}_I^\sharp(R_i) \subset \mathcal{U}_I^\sharp(R_{i+1})$, for all $i \in \mathbb{N}$. Therefore,

$$\rho(\mathcal{U}_I^\sharp) = \rho\left(\bigcup_{i=1}^\infty \mathcal{U}_I^\sharp(R_i)\right) = \lim_{i\to\infty} \rho(\mathcal{U}_I^\sharp(R_i)) \geq \lim_{i\to\infty}\left(1 - \frac{C_1}{R_i}\right) = 1.$$

Otherwise, we can write $\mathcal{U}_I^\sharp = \bigcap_{n=1}^\infty \bigcup_{i=1}^\infty \Pi_{J_n}^{-1}\mathcal{U}_{J_n}^\sharp(R_i)$, for any sequence $\{R_i\}_i$ with $R_0 \leq R_i \leq R_{i+1}$, for all $i \in \mathbb{N}$, $R_i \to \infty$ and for any sequence $\{J_n\}_n$ of compact subintervals of I such that $J_n \subset J_{n+1}$, for all $n \in \mathbb{N}$, and $\cup_n J_n = I$. Observe that, for all compact interval $J \subset I$ it is true that $\mathcal{U}_I^\sharp(R) \subset \Pi_J^{-1}\mathcal{U}_J^\sharp(R)$ so that $\rho(\Pi_J^{-1}\mathcal{U}_J^\sharp(R)) \geq 1 - C_1/R$. Moreover, it is easy to see the monotonicity properties

$$\bigcup_{i=1}^\infty \Pi_{J_n}^{-1}\mathcal{U}_{J_n}^\sharp(R_i) \supset \bigcup_{i=1}^\infty \Pi_{J_{n+1}}^{-1}\mathcal{U}_{J_{n+1}}^\sharp(R_i),$$

for all $n \in \mathbb{N}$, and

$$\Pi_{J_n}^{-1}\mathcal{U}_{J_n}^\sharp(R_i) \subset \Pi_{J_n}^{-1}\mathcal{U}_{J_n}^\sharp(R_{i+1}),$$

for all $n, i \in \mathbb{N}$. Therefore,

$$\begin{aligned}
\rho(\mathcal{U}_I^\sharp) = \rho\left(\bigcap_{n=1}^\infty \bigcup_{i=1}^\infty \Pi_{J_n}^{-1}\mathcal{U}_{J_n}^\sharp(R_i)\right) &= \lim_{n\to\infty} \rho\left(\bigcup_{i=1}^\infty \Pi_{J_n}^{-1}\mathcal{U}_{J_n}^\sharp(R_i)\right) \\
&= \lim_{n\to\infty}\lim_{i\to\infty} \rho(\Pi_{J_n}^{-1}\mathcal{U}_{J_n}^\sharp(R_i)) \geq \lim_{n\to\infty}\lim_{i\to\infty}\left(1 - \frac{C_1}{R_i}\right) = 1.
\end{aligned}$$

Thus, $\rho(\mathcal{U}_I^\sharp) = 1$.

It remains to prove that $t \mapsto \int_{\mathcal{Y}_I} |\mathbf{u}(t)|^2 d\rho(\mathbf{u})$ belongs to $L^\infty_{loc}(I)$. In that direction, we define an increasing sequence of cut-off functions $\{\phi_M\}_M$, that is, for all $M > 0$ we have that $\phi_M \in \mathcal{C}^\infty([0,\infty))$, $0 \leq \phi_M \leq 1$, $\phi_M(r) = 1$ for $0 \leq r \leq M$, $\phi(r) = 0$ for $r \geq 2M$ and $\phi_M \leq \phi_{M+1}$. Now, take any $t' \in I$ and observe that the function $f_{M,k}$, defined by $f_{M,k}(\mathbf{u}) = \phi_M(|P_k\mathbf{u}(t')|^2)|P_k\mathbf{u}(t')|^2$, where P_k is the Galerkin projector (see Section

2.1), belongs to $\mathcal{C}_b(\mathcal{Y}_I)$ and $f_{M,k}(\mathbf{u}) \leq |\mathbf{u}(t')|^2$, for all $\mathbf{u} \in \mathcal{Y}_I$. Then, from the convergence of ρ_{α_n} to ρ together with the hypothesis $\sup_{t\in I} \int_{\mathcal{Y}_I} |\mathbf{w}(t)|^2 d\rho_{\alpha_n}(\mathbf{w}) \leq C$, for all $n \in \mathbb{N}$, we obtain

$$\int_{\mathcal{Y}_I} f_{M,k}(\mathbf{u}) d\rho(\mathbf{u}) = \lim_{n\to\infty} \int_{\mathcal{Y}_I} f_{M,k}(\mathbf{u}) d\rho_{\alpha_n}(\mathbf{u}) \leq \limsup_{n\to\infty} \int_{\mathcal{Y}_I} |\mathbf{u}(t')|^2 d\rho_{\alpha_n}(\mathbf{u}) \leq C.$$

We can pass to the limit in the last inequality as $M \to \infty$ and, using the Monotone Convergence Theorem, we obtain that

$$\int_{\mathcal{Y}_I} |P_k \mathbf{u}(t')|^2 d\rho(\mathbf{u}) = \lim_{M\to\infty} \int_{\mathcal{Y}_I} f_{M,k}(\mathbf{u}) d\rho(\mathbf{u}) \leq C.$$

Again, using the Monotone Convergence Theorem, we can pass to the limit as $k \to \infty$ to find that

$$\int_{\mathcal{Y}_I} |\mathbf{u}(t')|^2 d\rho(\mathbf{u}) = \lim_{k\to\infty} \int_{\mathcal{Y}_I} |P_k \mathbf{u}(t')|^2 d\rho(\mathbf{u}) \leq C.$$

Now, since $t' \in I$ is arbitrary we obtain that $t \mapsto \int_{\mathcal{Y}_I} |\mathbf{u}(t)|^2 d\rho(\mathbf{u})$ belongs to $L^\infty(I)$. □

The previous result has a corresponding statement in terms of statistical solutions, which we write as follows.

Proposition 3.2. *Let I be any interval in $\mathbb{R}$ and let $\{\{\mu_t^\alpha\}_{t\in I}\}_{\alpha>0}$ be a family of α-Vishik-Fursikov statistical solutions over I, such that $\sup_{t\in I} \int_H |\mathbf{w}|^2 d\mu_t^\alpha(\mathbf{w}) \leq C$, for all $\alpha > 0$, for some constant $C \geq 0$. Then, there exists a sequence $\{\{\mu_t^{\alpha_n}\}_{t\in I}\}_n \subset \{\{\mu_t^\alpha\}_{t\in I}\}_{\alpha>0}$, with $\alpha_n \to 0$, converging to a Vishik-Fursikov statistical solution $\{\mu_t\}_{t\in \mathring{I}}$.*

Proof. By definition, for each $\alpha > 0$, there exists an α-Vishik-Fursikov measure ρ_α over I such that $\mu_t^\alpha = \Pi_t \rho_\alpha$, for all $t \in I$. Observe that the family $\{\rho_\alpha\}_{\alpha>0}$ fulfills the hypothesis of Proposition 3.1, then there exists a sequence $\{\rho_{\alpha_n}\}_n \subset \{\rho_\alpha\}_{\alpha>0}$, with $\alpha_n \to 0$, that converges to a Borel probability measure ρ in $\mathcal{Y}_I$ such that $\Pi_{\mathring{I}}\rho$ is a Vishik-Fursikov measure over $\mathring{I}$. Thus, as a simple consequence of the Change of Variables Theorem (see Section 2.2), we obtain the convergence of the α_n-Vishik-Fursikov statistical solutions $\{\Pi_t \rho_{\alpha_n}\}_{t\in I}$ to the Vishik-Fursikov statistical solution $\{\Pi_t \rho\}_{t\in\mathring{I}}$. In other words, $\{\{\mu_t^{\alpha_n}\}_{t\in I}\}_k$ converges to the Vishik-Fursikov statistical solution $\{\mu_t\}_{t\in\mathring{I}}$, where $\mu_t = \Pi_t\rho$, for all $t \in \mathring{I}$. □

A different way of obtaining the uniform tightness condition in the family of α-Vishik-Fursikov measure is to assume that the interval I is bounded and closed on the left and impose uniform boundedness on the initial mean kinetic energy over the α-Vishik-Fursikov measures. This is done in the next result:

Corollary 3.1. *Let I be an interval in $\mathbb{R}$ which is bounded and closed on the left, with left end point t_0. Let $\{\rho_\alpha\}_{\alpha>0}$ be a family of α-Vishik-Fursikov measures over I such that $\int_{\mathcal{Y}_I} |\mathbf{w}(t_0)|^2 d\rho_\alpha(\mathbf{w}) \leq C$, for all $\alpha > 0$, for some constant $C \geq 0$. Then, there exists a sequence $\{\rho_{\alpha_n}\}_n \subset \{\rho_\alpha\}_{\alpha>0}$, with $\alpha_n \to 0$, that converges to a Borel probability measure ρ in $\mathcal{Y}_I$ such that ρ is carried by $\mathcal{U}_I^\sharp$ and $\Pi_{\mathring{I}}\rho$ is a Vishik-Fursikov measure over $\mathring{I}$.*

Proof. We only need to check that the family $\{\rho_\alpha\}$ fulfills the hypothesis of Proposition 3.1. In order to do so, observe that for all $\mathbf{w} \in \mathcal{U}_I^\alpha$ and for all $t \in I$ with $t \geq t_0$,

$$|\mathbf{w}(t)|^2 \leq |\mathbf{w}(t_0)|^2 e^{-\lambda_1 \nu (t-t_0)} + \frac{1}{\lambda_1^2 \nu^2} \|\mathbf{f}\|^2_{L^\infty(t_0,t;H)} (1 - e^{-\lambda_1 \nu (t-t_0)}),$$

therefore

$$\int_{\mathcal{U}_I^\alpha} |\mathbf{w}(t)|^2 d\rho_\alpha(\mathbf{w}) \leq \int_{\mathcal{U}_I^\alpha} |\mathbf{w}(t_0)|^2 d\rho_\alpha(\mathbf{w}) + \frac{1}{\lambda_1^2 \nu^2} \|\mathbf{f}\|^2_{L^\infty(t_0,t;H)}.$$

Since by hypothesis $\int_{\mathcal{Y}_I} |\mathbf{w}(t_0)|^2 d\rho_\alpha(\mathbf{w}) \leq C$, for all $\alpha > 0$, and ρ_α is carried by $\mathcal{U}_I^\alpha$, then

$$\sup_{\alpha>0} \sup_{t\in I} \int_{\mathcal{Y}_I} |\mathbf{w}(t)|^2 d\rho_\alpha(\mathbf{u}) \leq C_1,$$

where $C_1 = C + 1/(\nu^2\lambda_1^2)\|f\|^2_{L^\infty(I,H)}$. □

As before, the previous result has a corresponding statement in terms of statistical solutions, which we write as follows.

Corollary 3.2. *Let I be any interval in $\mathbb{R}$ and let $\{\{\mu_t^\alpha\}_{t\in I}\}_{\alpha>0}$ be a family of α-Vishik-Fursikov statistical solutions over I, such that $\int_H |\mathbf{w}|^2 d\mu_{t_0}^\alpha(\mathbf{w}) \leq C$, for all $\alpha > 0$, for some constant $C \geq 0$. Then, there exists a sequence $\{\{\mu_t^{\alpha_n}\}_{t\in I}\}_n \subset \{\{\mu_t^\alpha\}_{t\in I}\}_{\alpha>0}$, with $\alpha_n \to 0$, that converges to a Vishik-Fursikov statistical solution $\{\mu_t\}_{t\in \mathring{I}}$.*

In the last two results we obtain a limit of α-Vishik-Fursikov measures over an interval I which is a Vishik-Fursikov measure only in the interior of the interval I. If I is an interval open on the left then the limit is in fact a Vishik-Fursikov measure over the whole interval I. The problem is when I is closed and bounded on the left, since in this case we may lose the continuity at the left end point of I of the strengthened mean kinetic energy for the limit measure (see condition *(iii)* of the Definition 2.4). This problem is, in fact, inherited from an analogous problem for individual weak solutions, as described in Section 2.5, and which led us to introduce the spaces $\mathcal{U}_I^\sharp(R)$ and $\mathcal{U}_I^\sharp$.

In the next result we impose some conditions over a family of α-Vishik-Fursikov measures in order to have that the limit is in fact a Vishik-Fursikov measure over the whole interval I.

In fact we impose conditions only on the initial measures $\Pi_{t_0}\rho_\alpha$, where t_0 is the left end point of the interval I. These initial measures shall converge in a slightly stronger sense to a measure μ_0 in H_w. Since the σ-algebra of Borel sets with respect to the weak topology of H coincides with that with respect to the strong topology we may consider μ_0 as a Borel probability either on H or on H_w.

Theorem 3.1. *Let I be an interval in $\mathbb{R}$ bounded and closed on the left, with left end point t_0, and let $\{\rho_\alpha\}_{\alpha>0}$ be a family of α-Vishik-Fursikov measures over I. In addition, suppose that there exists a Borel probability measure μ_0 on H such that*

(i) $\int_H |\mathbf{u}|^2 d\mu_0(\mathbf{u}) < \infty$;

(ii) $\Pi_{t_0}\rho_\alpha \overset{*}{\rightharpoonup} \mu_0$ *in* $\mathcal{P}(H_w)$;

(iii) and for all $\psi \in \Psi$

$$\lim_{\alpha\to 0} \int_{\mathcal{Y}_I} \psi(|\mathbf{u}(t_0)|^2) d\rho_\alpha(\mathbf{u}) = \int_H \psi(|\mathbf{u}|^2) d\mu_0(\mathbf{u}).$$

Then, there exists a sequence $\{\rho_{\alpha_n}\}_n \subset \{\rho_\alpha\}_{\alpha>0}$, with $\alpha_n \to 0$, that converges to a Borel probability measure ρ in $\mathcal{Y}_I$ such that ρ is a Vishik-Fursikov measure over I and $\Pi_{t_0}\rho = \mu_0$.

Proof. Observe that by taking $\psi \equiv 1$ we can see that the family $\{\rho_\alpha\}_{\alpha>0}$ fulfills the hypothesis of Corollary 3.1. Thus, there exist a sequence $\{\rho_{\alpha_n}\}_n \subset \{\rho_\alpha\}_{\alpha>0}$, with $\alpha_n \to 0$, and Borel probability measure ρ in $\mathcal{Y}_I$ such that $\rho_{\alpha_n} \overset{*}{\rightharpoonup} \rho$ in $\mathcal{P}(\mathcal{Y}_I)$, as $n \to \infty$, ρ is carried by $\mathcal{U}_I^\sharp$, and $\Pi_{\mathring{I}}\rho$ is a Vishik-Fursikov measure over $\mathring{I}$. Then, in order to obtain that ρ is a Vishik-Fursikov measure over I, it remains to prove that for all $\psi \in \Psi$ it holds true that

$$\lim_{t\to t_0^+} \int_{\mathcal{U}_I^\sharp} \psi(|\mathbf{u}(t)|^2) d\rho(\mathbf{u}) = \int_{\mathcal{U}_I^\sharp} \psi(|\mathbf{u}(t_0)|^2) d\rho(\mathbf{u}). \tag{40}$$

Observe that for any $\phi \in \mathcal{C}_b(H_w)$ the function defined by $\varphi = \phi \circ \Pi_{t_0}|_{\mathcal{Y}_I}$ belongs to $\varphi \in \mathcal{C}_b(\mathcal{Y}_I)$. Then, it is straightforward from the convergence of ρ_{α_n}, from hypothesis (ii) and from the Change of Variables Theorem that

$$\int_H \phi(\mathbf{u}) d\mu_0(\mathbf{u}) = \int_{\mathcal{Y}_I} \phi(\mathbf{u}(t_0)) d\rho(\mathbf{u}), \text{ for all } \phi \in \mathcal{C}_b(H_w).$$

Therefore, since H_w is a completely regular Hausdorff space and $\mu_0, \Pi_{t_0}\rho \in \mathcal{P}(H_w; t)$ we obtain that $\Pi_{t_0}\rho = \mu_0$ (see (5)).

Now, take $\psi \in \Psi$ and observe that, since any $\mathbf{u} \in \mathcal{U}_I^\sharp$ is weakly continuous at t_0 and ψ is nondecreasing and continuous, then $\psi(|\mathbf{u}(t_0)|^2) \leq \liminf_{t\to t_0} \psi(|\mathbf{u}(t)|^2)$. Therefore, it follows from Fatou's Lemma that

$$\int_{\mathcal{U}_I^\sharp} \psi(|\mathbf{u}(t_0)|^2) d\rho(\mathbf{u}) \leq \liminf_{t\to t_0} \int_{\mathcal{U}_I^\sharp} \psi(|\mathbf{u}(t)|^2) d\rho(\mathbf{u}). \tag{41}$$

Let ϕ_M be a function in $\mathcal{C}^1([0,\infty))$ defined as $\phi_M(r) = 1$, for all $0 \leq r \leq M$, $\phi_M(r) = 0$, for all $r \geq 2M$, $\phi_M \leq \phi_{M+1}$, and $0 \leq \phi_M \leq 1$, for all $M \in \mathbb{N}$. It is clear that $\varphi_m(\mathbf{u}) := \psi(|P_m\mathbf{u}(t)|^2)\phi_M(|P_m\mathbf{u}(t)|^2)$ belongs to $\mathcal{C}_b(\mathcal{Y}_I)$, for any $t \in I$, so that

$$\int_{\mathcal{Y}_I} \psi(|P_m\mathbf{u}(t)|^2)\phi_M(|P_m\mathbf{u}(t)|^2) d\rho(\mathbf{u}) = \lim_{n\to\infty} \int_{\mathcal{Y}_I} \psi(|P_m\mathbf{u}(t)|^2)\phi_M(|P_m\mathbf{u}(t)|^2) d\rho_{\alpha_n}(\mathbf{u}).$$

Since $\phi_M \leq 1$ and $\psi(|P_m\mathbf{u}(t)|^2) \leq \psi(|\mathbf{u}(t)|^2)$ we find that the right hand side of the last identity is bounded above by $\limsup_{n\to\infty} \int_{\mathcal{Y}_I} \psi(|\mathbf{u}(t)|^2) d\rho_{\alpha_n}(\mathbf{u})$. On the other hand, using the Monotone Convergence Theorem twice we obtain that

$$\lim_{m\to\infty} \lim_{M\to\infty} \int_{\mathcal{Y}_I} \psi(|P_m\mathbf{u}(t)|^2)\phi_M(|P_m\mathbf{u}(t)|^2) d\rho(\mathbf{u}) = \int_{\mathcal{Y}_I} \psi(|\mathbf{u}(t)|^2) d\rho(\mathbf{u}).$$

Therefore, for any $t \in I$,

$$\int_{\mathcal{Y}_I} \psi(|\mathbf{u}(t)|^2) d\rho(\mathbf{u}) \leq \limsup_{n\to\infty} \int_{\mathcal{Y}_I} \psi(|\mathbf{u}(t)|^2) d\rho_{\alpha_n}(\mathbf{u}). \tag{42}$$

Now, using inequality (42), estimate (17) and the facts that ρ_α is carried by $\mathcal{U}_I^\alpha$ and ψ is nondecreasing, we obtain

$$\begin{aligned}
&\limsup_{t\to t_0} \int_{\mathcal{Y}_I} \psi(|\mathbf{u}(t)|^2) d\rho(\mathbf{u}) \leq \limsup_{t\to t_0} \limsup_{n\to\infty} \int_{\mathcal{Y}_I} \psi(|\mathbf{u}(t)|^2) d\rho_{\alpha_n}(\mathbf{u}) \\
&\leq \limsup_{t\to t_0} \limsup_{n\to\infty} \left(\int_{\mathcal{Y}_I} \psi(|\mathbf{u}(t_0)|^2) d\rho_{\alpha_n}(\mathbf{u}) + \frac{1}{\lambda_1 \nu} \|f\|_{L^\infty(I,H)} \sup_{r\geq 0} \psi'(r)(t-t_0) \right) \\
&\leq \limsup_{n\to\infty} \int_{\mathcal{Y}_I} \psi(|\mathbf{u}(t_0)|^2) d\rho_{\alpha_n}(\mathbf{u}) \leq \int_H \psi(|\mathbf{u}|^2) d\mu_0(\mathbf{u}),
\end{aligned}$$

where the last inequality follows from hypothesis (iii).

To conclude, we use that $\Pi_{t_0}\rho = \mu_0$ in the last inequality together with (41) to obtain that (40) holds true. $\square$

Now, we write the previous result in terms of statistical solutions.

Theorem 3.2. *Let I be an interval in $\mathbb{R}$ bounded and closed on the left, with left end point t_0, and let $\{\{\mu_t^\alpha\}_{t\in I}\}_{\alpha>0}$ be a family of α-Vishik-Fursikov statistical solutions over I. In addition, suppose that there exists a Borel probability measure μ_0 on H such that*

(i) $\int_H |\mathbf{u}|^2 d\mu_0(\mathbf{u}) < \infty$;

(ii) $\mu^\alpha_{t_0} \overset{*}{\rightharpoonup} \mu_0$ *in* $\mathcal{P}(H_w)$*;*

(iii) and for all $\psi \in \Psi$

$$\lim_{\alpha\to 0}\int_H \psi(|\mathbf{u}|^2)d\mu^\alpha_{t_0}(\mathbf{u}) = \int_H \psi(|\mathbf{u}|^2)d\mu_0(\mathbf{u}).$$

Then, there exists a sequence $\{\{\mu^{\alpha_n}_t\}_{t\in I}\}_n \subset \{\{\mu^\alpha_t\}_{t\in I}\}_{\alpha>0}$*, with* $\alpha_n \to 0$*, that converges to a Vishik-Fursikov statistical solution* $\{\mu_t\}_{t\in I}$ *such that* $\mu_{t_0} = \mu_0$*.*

Proof. It follows from the definition of α-Vishik-Fursikov statistical solution the existence of an α-Vishik-Fursikov measure ρ_α over I such that $\mu^\alpha_t = \Pi_t\rho_\alpha$, for all $t \in I$. It is clear that the family $\{\rho_\alpha\}_{\alpha>0}$ fulfills the hypothesis of Theorem 3.1 so that there exists a sequence $\{\rho_{\alpha_n}\}_n \subset \{\rho_\alpha\}_{\alpha>0}$ that converges to a Vishik-Fursikov measure ρ over I. Therefore, the corresponding sequence $\{\{\mu^{\alpha_n}_t\}_{t\in I}\}_n$ converges to the Vishik-Fursikov statistical solution $\{\mu_t\}_{t\in I}$, where $\mu_t = \Pi_t\rho$, for all $t \in I$. □

A case of particular interest for approximation purposes is when the measures ρ_α have the same initial projection $\Pi_{t_0}\rho_\alpha = \mu_0$, for all α. This leads us to the following two corollaries, in terms of Vishik-Fursikov measures and Vishik-Fursikov statistical solutions, respectively.

Corollary 3.3. *Let* I *be an interval in* $\mathbb{R}$ *closed and bounded on the left, with left end point* t_0*, and* μ_0 *a Borel probability on* H *such that* $\int_H |\mathbf{u}|^2 d\mu_0(\mathbf{u}) < \infty$*. Let* $\{\rho_\alpha\}_{\alpha>0}$ *be a family of* α*-Vishik-Fursikov measures over* I *such that* $\Pi_{t_0}\rho_\alpha = \mu_0$*, for all* $\alpha > 0$*. Then, there exists a sequence* $\{\rho_{\alpha_n}\}_n \subset \{\rho_\alpha\}_{\alpha>0}$*, with* $\alpha_n \to 0$*, that converges to a Borel probability measure* ρ *in* $\mathcal{Y}_I$ *such that* ρ *is a Vishik-Fursikov measure over* I *and* $\Pi_{t_0}\rho = \mu_0$*.*

Corollary 3.4. *Let* I *be an interval in* $\mathbb{R}$ *closed and bounded on the left, with left end point* t_0*, and* μ_0 *a Borel probability on* H *such that* $\int_H |\mathbf{u}|^2 d\mu_0(\mathbf{u}) < \infty$*. Let* $\{\{\mu^\alpha_t\}_{t\in I}\}_{\alpha>0}$ *be a family of* α*-Vishik-Fursikov statistical solutions over* I *such that* $\mu^\alpha_{t_0} = \mu_0$*, for all* $\alpha > 0$*. Then, there exists a sequence* $\{\{\Pi_t\rho_{\alpha_n}\}_{t\in I}\}_n \subset \{\{\mu^\alpha_t\}_{t\in I}\}_{\alpha>0}$*, with* $\alpha_n \to 0$*, that converges to a Vishik-Fursikov statistical solution* $\{\mu_t\}_{t\in I}$ *such that* $\mu_{t_0} = \mu_0$*.*

3.2 Stationary statistical solution

We say that a family $\{\mu_\alpha\}_{\alpha>0}$ of stationary α-Vishik-Fursikov statistical solutions converges to a stationary Vishik-Fursikov statistical solution if there exists a Borel probability measure μ on H such that μ is a stationary Vishik-Fursikov statistical solution and

$$\lim_{\alpha\to 0}\int_H \phi(\mathbf{u})\mathrm{d}\mu_\alpha(\mathbf{u}) = \int_H \phi(\mathbf{u})\mathrm{d}\mu(\mathbf{u}),$$

for all $\phi \in \mathcal{C}_b(H_w)$.

Theorem 3.3. *Let* $\{\mu_\alpha\}_{\alpha>0}$ *be a family such that, for each* $\alpha > 0$*,* μ_α *is a stationary* α*-Vishik-Fursikov statistical solution. Suppose that* $\int_H |\mathbf{u}|^2 d\mu_\alpha(\mathbf{u}) \leq C$*, for all* $\alpha > 0$*, for some* $C \geq 0$*. Then, there exists a sequence* $\{\mu_{\alpha_n}\}_n \subset \{\mu_\alpha\}_{\alpha>0}$*, with* $\alpha_n \to 0$*, that converges to a Borel probability measure* μ *on* H*. Moreover,* μ *is a stationary Vishik-Fursikov statistical solution.*

Proof. Since μ_α is a stationary α-Vishik-Fursikov statistical solution, for each $\alpha > 0$, there exists an invariant Vishik-Fursikov measure ρ_α over an interval I unbounded on the right, such that $\mu_\alpha = \Pi_t\rho_\alpha$, at any time $t \in I$. Hence, using the Change of Variables Theorem it follows that

$$\int_{\mathcal{Y}_I} |\mathbf{u}(t)|^2 \mathrm{d}\rho_\alpha(\mathbf{u}) = \int_H |\mathbf{u}|^2 \mathrm{d}\mu_\alpha(\mathbf{u}),\ \forall t \in I,\ \forall \alpha > 0.$$

From the hypothesis $\int_H |\mathbf{u}|^2 \mathrm{d}\mu_\alpha(\mathbf{u}) \leq C$, for all $\alpha > 0$, and the last identity, we obtain that

$$\sup_{t \in I} \int_{\mathcal{Y}_I} |\mathbf{u}(t)|^2 \mathrm{d}\rho_\alpha(\mathbf{u}) \leq C, \ \forall \alpha > 0.$$

Thus, we can apply Proposition 3.1 to obtain a sequence, $\{\rho_{\alpha_n}\}_n \subset \{\rho_\alpha\}_{\alpha>0}$, with $\alpha_n \to 0$, and a Vishik-Fursikov measure ρ over I such that $\rho_{\alpha_n} \overset{*}{\rightharpoonup} \rho$ in $\mathcal{P}(\mathcal{Y}_I)$. First, let us check that ρ is an invariant Vishik-Fursikov measure, which by (5) is equivalent to show that

$$\int_{\mathcal{Y}_I} \varphi(\sigma_\tau \mathbf{u}) \mathrm{d}\rho(\mathbf{u}) = \int_{\mathcal{Y}_I} \varphi(\mathbf{u}) \mathrm{d}\rho(\mathbf{u}), \text{ for all } \varphi \in \mathcal{C}_b(\mathcal{Y}_I).$$

We already have that for all $\varphi \in \mathcal{C}_b(\mathcal{Y}_I)$,

$$\int_{\mathcal{Y}_I} \varphi(\sigma_\tau \mathbf{u}) \mathrm{d}\rho_{\alpha_n}(\mathbf{u}) = \int_{\mathcal{Y}_I} \varphi(\mathbf{u}) \mathrm{d}\rho_{\alpha_n}(\mathbf{u}).$$

Note also that $\varphi \circ \sigma_\tau$ belongs to $\mathcal{C}_b(\mathcal{Y}_I)$ for all $\varphi \in \mathcal{C}_b(\mathcal{Y}_I)$, so that, since $\rho_{\alpha_n} \overset{*}{\rightharpoonup} \rho$ in $\mathcal{P}(\mathcal{Y}_I)$, we obtain that

$$\int_{\mathcal{Y}_I} \varphi(\sigma_\tau \mathbf{u}) \mathrm{d}\rho(\mathbf{u}) = \lim_{n\to\infty} \int_{\mathcal{Y}_I} \varphi(\sigma_\tau \mathbf{u}) \mathrm{d}\rho_{\alpha_n}(\mathbf{u}) = \lim_{n\to\infty} \int_{\mathcal{Y}_I} \varphi(\mathbf{u}) \mathrm{d}\rho_{\alpha_n}(\mathbf{u}) = \int_{\mathcal{Y}_I} \varphi(\mathbf{u}) \mathrm{d}\rho(\mathbf{u}).$$

Define $\mu := \Pi_t \rho$, for an arbitrary time $t \in I$. Then, μ is a stationary Vishik-Fursikov statistical solution by definition. Moreover, $\mu_{\alpha_n} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{P}(H_w)$. Indeed, let $\phi \in \mathcal{C}_b(H_w)$, then $\varphi := \phi \circ \Pi_t$ belongs to $\mathcal{C}_b(\mathcal{Y}_I)$ and

$$\lim_{n\to\infty} \int_H \phi(\mathbf{u}) \mathrm{d}\mu_{\alpha_n}(\mathbf{u}) = \lim_{n\to\infty} \int_{\mathcal{Y}_I} \varphi(\mathbf{u}) \mathrm{d}\rho_{\alpha_n}(\mathbf{u}) = \int_{\mathcal{Y}_I} \varphi(\mathbf{u}) \mathrm{d}\rho(\mathbf{u}) = \int_H \phi(\mathbf{u}) \mathrm{d}\mu(\mathbf{u}).$$

This concludes the proof. □

4 Conclusions

We have proved that, under natural conditions, families of statistical solutions of the 3D Navier-Stokes-α model, depending on the parameter α, possess, as α goes to zero, subsequences that converge to a statistical solution of the 3D Navier-Stokes equations. The main condition for the convergence is that the mean kinetic energy of the family of statistical solutions be uniformly bounded. This yields suitable tightness and compactness properties needed for the existence of a convergent subsequence.

The statistical solutions contain the statistical information of a given flow, including turbulent flows, which are the case of most interest. It is therefore natural to study conditions that guarantee that the statistical information obtained from an approximate problem converges to the original problem in a suitable sense. We succeeded in proving this convergence under simple and natural conditions. This result implies that the statistical information obtained from the 3D Navier-Stokes-α model are good approximations of the statistical information of flows modelled by the 3D Navier-Stokes equations.

Moreover, the techniques developed in this paper seem to allow for an extension of the result to a wide range of models and approximations. This is currently a work in progress and will be presented elsewhere.

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